JCARM
Journal of Contemporary Academic Research and Methodologies

An Online Peer Reviewed / Refereed Journal
Volume 1 | Issue 3 | April 2026
Website: www.journal.jcarm.in

Research

A Computational Study of Commutator-Driven Iterative Dynamics on the Heisenberg Group with Affine Growth Behavior

KufreAbasi E. Essien[1]*, Imoh S. Abraham[1], Louis E. Akpabio[1], Ukeme A. Umoh[1], Promise Agbontaen[1], Nsemeke Eyo[1]

[1]Department of Physics, University of Uyo, Nigeria.

Correspondence should be addressed to: kantville@gmail.com

Abstract: We present a computational investigation of a commutator-driven iterative dynamical system defined on the Heisenberg group. The scheme is constructed by repeatedly updating a group element through multiplication with its commutator relative to a fixed element, thereby inducing a nonlinear evolution governed by the underlying non-abelian structure. Using a minimal implementation in Python, we generate trajectories from prescribed initial conditions and examine their evolution over discrete iterations. Our numerical experiments reveal a consistent and robust linear growth pattern in the group norm of the iterates, characterized by an affine growth. This behavior persists across multiple initial configurations, indicating that the observed dynamics are not incidental but intrinsic to the commutator-driven update mechanism. Further analysis shows that the growth is primarily driven by accumulation in the central component of the Heisenberg group, reflecting the non-commutative coupling embedded in the group law. The results highlight a clear departure from classical contractive or fixed-point iterative schemes, instead demonstrating a deterministic growth regime arising from algebraic structure. The simplicity of the computational framework, combined with the clarity of the emergent linear law, makes this system a useful prototype for exploring nonlinear dynamics on non-abelian groups. These findings open pathways for extending commutator-based iterative methods to more general Lie groups and for investigating their potential applications in computational and theoretical physics.

Keywords: Affine growth, Heisenberg group, non-commutative coupling, nonlinear dynamics, numerical experiments.

1. Introduction

https://doi.org/10.5281/zenodo.20282253

Iterative methods play a central role in modern computational mathematics and physics, particularly in the study of nonlinear equations, fixed point problems, and dynamical systems arising in applied sciences. Classical frameworks such as Mann-type iterations have been extensively studied for their convergence properties in Banach and Hilbert spaces, forming a foundational basis for nonlinear operator theory and computational fixed point analysis (Mann, 1953; Browder, 1965; Baillon, 1975; Goebel & Kirk, 1990; Xu, 2002; Chidume, 2009). These schemes typically rely on contractive or nonexpansive mappings, where convergence is guaranteed under appropriate metric or convexity assumptions (Bauschke & Combettes, 2017; Saqib *et al.*, 2015; Saqib *et al.,* 2026).

Recent advances in iterative algorithms have extended these ideas to more complex nonlinear settings, including variational inequalities, semigroup dynamics, and applications in engineering and applied sciences (Mohamadi, 2011; Dehaish et al., 2015; Ahmad et al., 2026; Alharthi et al., 2026). Despite this progress, most existing methods remain embedded in linear or convex frameworks, limiting their applicability to systems with intrinsic non-commutative structure.

Non-abelian groups, particularly nilpotent Lie groups such as the Heisenberg group, provide a natural environment for extending iterative dynamics beyond classical Euclidean settings. The Heisenberg group, a fundamental example in group theory and mathematical physics, exhibits a non-trivial commutator structure that fundamentally alters dynamical behavior (Hall, 1934; Robinson, 2004; Shah & Noor, 2024; Akran *et al.,* 2025). Its algebraic properties make it a canonical model for studying nonlinear evolution driven by non-commutativity.

Furthermore, results from geometric group theory suggest that algebraic structure strongly influences asymptotic growth behavior, especially in nilpotent groups where structured growth laws may emerge (Stoll, 1996; Shumyatsky, 2005; Kirk & Sims, 2001; Okorie *et al.,* 2018). However, the explicit computational study of iterative schemes driven purely by commutator dynamics remains largely unexplored.

Motivated by this research gap, this work introduces a commutator-driven iterative process on the Heisenberg group and investigates its computational behavior. Unlike classical fixed-point iterations, the proposed scheme generates a nonlinear trajectory governed by internal group interactions rather than external contraction mechanisms.

The principal finding of this study is the emergence of a linear growth law in the group norm, of the form y=mx+c, with m > 0 and c ≥ 0. This behavior is shown to arise from systematic accumulation in the central component of the group, reflecting the intrinsic non-commutative structure. The study thus provides new insight into how algebraic properties can directly generate macroscopic dynamical laws.

2. Methods

2.1 Heisenberg Group Information

We begin by constructing the Heisenberg group as a computational space. The system upon which our computations are performed is therefore defined on the Heisenberg group denoted by H, and realized on R^3 with elements represented as triples:

$$x = \left(x_1, \ x_2, x_3\right) \qquad (1)$$

$$a = \left(a_1, \ a_2, a_3\right) \qquad (2)$$

For ease of classification and functionality of the group in our computations, we define the following for the group using the group elements.

The identity element is defined as:

$$e = (0, \ 0, \ 0) \qquad (3)$$

The group multiplication is defined as:

$$x.a = \left(x_1 + a_1, \ x_2 + a_2, \ x_3 + a_3 + x_1 a_3\right) \qquad (4)$$

The inverse map of the group is defined as:

$$x^{-1} = \left(- x_1, - x_2, - x_3 + - x_1 - x_2\right) \text{..............}(5)$$

The commutator is defined as:

$$[x, a] = x^{-1} a^{-1} x a \text{.............} (6)$$

This structure defines a non-abelian nilpotent Lie group whose non-commutativity is fully captured by the central coordinate, and is characterized by a non-commutative multiplication law (Hall, 1934; Robinson, 2004).

2.2 Operator Formulation

For $x, a \in = H$, we introduce a commutator–driven nonlinear operator defined as:

$$T_a : \ H \longrightarrow H, \qquad (7a)$$

$$T_a(x) = \ x.[x, a] \qquad (7b)$$

Thus, dynamical system studied in this work is the discrete–time evolution given as:

$$x_{n+1}: T_a\left(x_n\right), \quad x_0 \in H, \quad n \in N \qquad (8)$$

Therefore, the numerical scheme in our programs iterates:

$$x_{n+1} = x_{n}\cdot\left[x_n, a\right], \quad n \geq 0 \qquad (9)$$

This formulation departs from the classical iterative schemes, which are typically designed to ensure convergence via nonexpansive mappings (Browder, 1965; Xu, 2002). Unlike the classical fixed-point iterations, T_a is not contractive in general and it does not arise for convex averaging. Instead, it is an intrinsically linear operator generated by the internal commutator structure of the group.

2.3 Structural form of the iteration.

Using the Heisenberg group law, the commutator can be explicitly computed, yielding a central contribution depending only on the first two coordinates. Consequently, the iteration admits the structural decomposition:

$$x_{n+1} = \left(x_{1,n}, x_{2,n}, x_{3,n}\right) + \phi\left(x_n, a\right) \qquad (10)$$

accumaltes nonlinear interaction terms; and growth behaviour is governed by central extension dynamics, $\phi(x_n, a)$.

2.4 Norm functional, growth characterization and data analysis

To quantify the dynamic evolution, we define the group norm functional as:

$$||x|| = \left|x_1\right| + \left|x_2\right| + \left|x_3\right| \qquad (11)$$

The discrete-time observable is therefore given as :

$$y_n = \left|\left|x_n\right|\right| \qquad (12)$$

This scalar reduction enables direct comparison with classical growth laws and therefore facilitates regression-based analysis. Empirically, the sequence y_n exhibits affine scaling:

$$y_n = mn + c, \quad m > 0, \quad c > 0 \qquad (13)$$

In our programs, the trapectory data $(n_{,}|x_n|)$ were recored and analysed using linear regression and correlative techniques commonly emphyed in computational studies of iterative systems (Grau-Sánchet *et al.,* 2015; Zhang *et al.,* 2021). In the trapectory data, n is the iteration index while $|x_n|$ is the group norm.

2.5 Post processing of data

From the computational experiment, the generated data: $\left(n_{,} y_{n}\right) = (n_{,}|\left|x_{n}\right||)$ are were fitted to the model, $y = mx + c$; where $x = n$(iteration index) $y = |x_{n}|$ (group norm) and $c =$ initial offset.

Linear regression analysis was performed on the data using the least-squares method which minimizes: $\sum(y_{i} - \left(mx_{i} + c\right))^2$ for all data points i in the data set. The coefficients m and c for N data points, x_{i}, y_{i}, were computed using the formulas:

$$m = (N\sum x_{i}y_{i} - \left(\sum x_{i}\right)\left(\sum y_{i}\right))/ (N\sum x_{i}^{2} - \left(\sum x_{i}\right))^{2} \qquad (14)$$

$$c = (\sum y_{i} - \frac{m\sum x_{i})}{N} \qquad (15)$$

To test how well the regression line fits the data, a goodness of fit was evaluated using the coefficeint of determination, R^{2} given as:

$$R^{2} = 1 - \frac{SS_{res}}{SS_{tot}} \qquad (16)$$

SS_{res} = residual sum of squares

$$= \sum_{i=1}^{N}(y_{i} - \hat{y}_{i})^{2} \qquad (17)$$

SS_{tot} = total sum of squares

$$= \sum_{i=1}^{N}(y_{i} - \tilde{y})^{2} \qquad (18)$$

Where y_{i} = observed data value, $\hat{y} = mx_{i} + c$is the predicted value from the regression line; and $\tilde{y}$ = mean of the observed values, evaluated as: $\frac{1}{N}\sum_{i=1}^{N} y_{i}$.

2.6 Computational Experiment and Data Processing

The primary computational objective was to determine whether the commutator-driven iteration will exhibit bounded dynamics (stability), sublinear growth, or structured divergence for our cconstructed dynamical system.

The Operation $T_{a\,(x)}$ was implemented using a minimal Python framework based on explicit algebraic expressions that executed the numerical formulation of the dynamical system, namely: group multiplication, inverse evaluation and commutator construction.

Simulations were performed for a fixed member of iterations N = 20, with representative initial conditions $x_0 = (1, 1, 0)$, a = (0,1,0). At each iteration, the scalar norm y_n was computed and stored within our program enviroment within in a structured data file of the form (n, y_n), where n is the iteration index.

The resulting dataset was post-processed for visaulization and linear regression analysis. GNUplot was used as an external plotting tool for data visualization.

3. Results and Discussion

3.1 Group Norm Evolution

The discrete-time evolution of the group norm was obtained by plotting the trajectory data $(n, |x_n|)$. The result is the straight line graph shown in Figure 3.1.

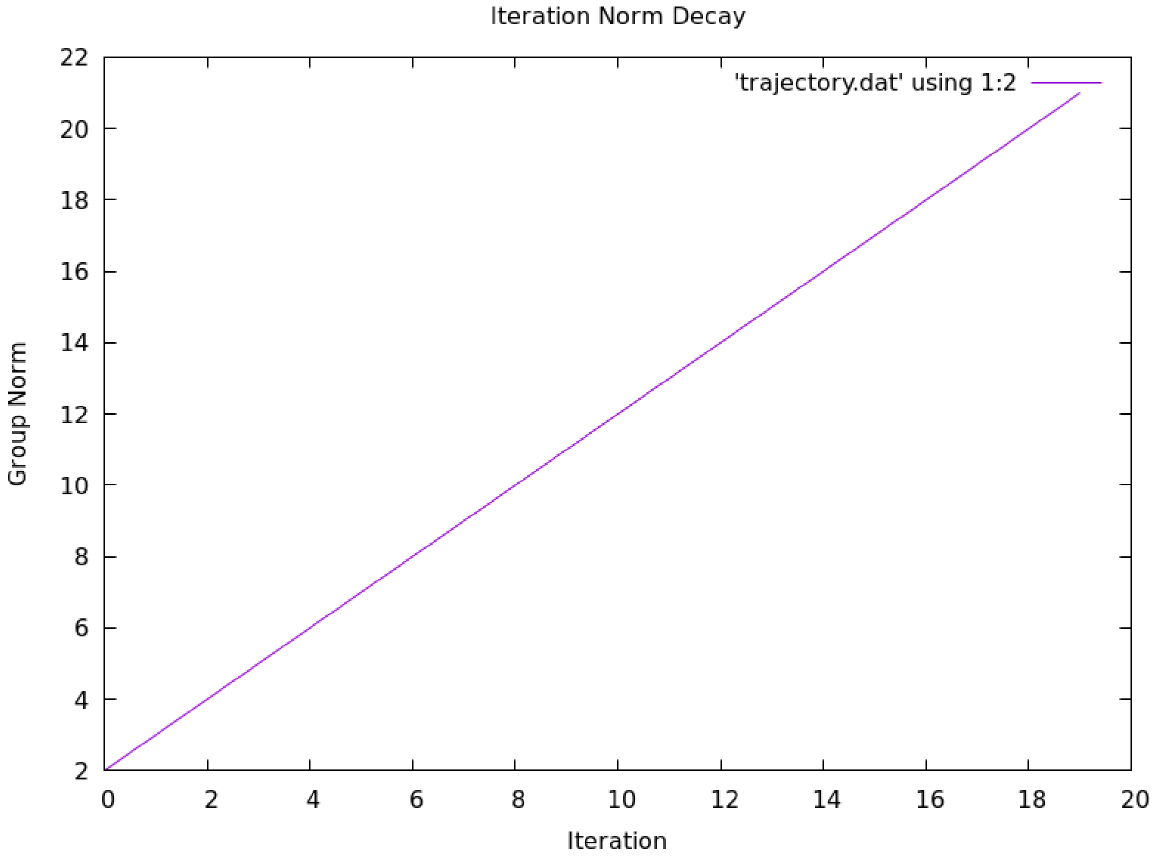

Figure 3.1: Trajectory of group norm

3.2 Linear regression analysis and goodness of fit test

Using the least-squares method, linear regression analysis was performed on the trajectory data $(n, |x_n|)$. The data was fitted to the regression line and the result is shown in Figure 3.2

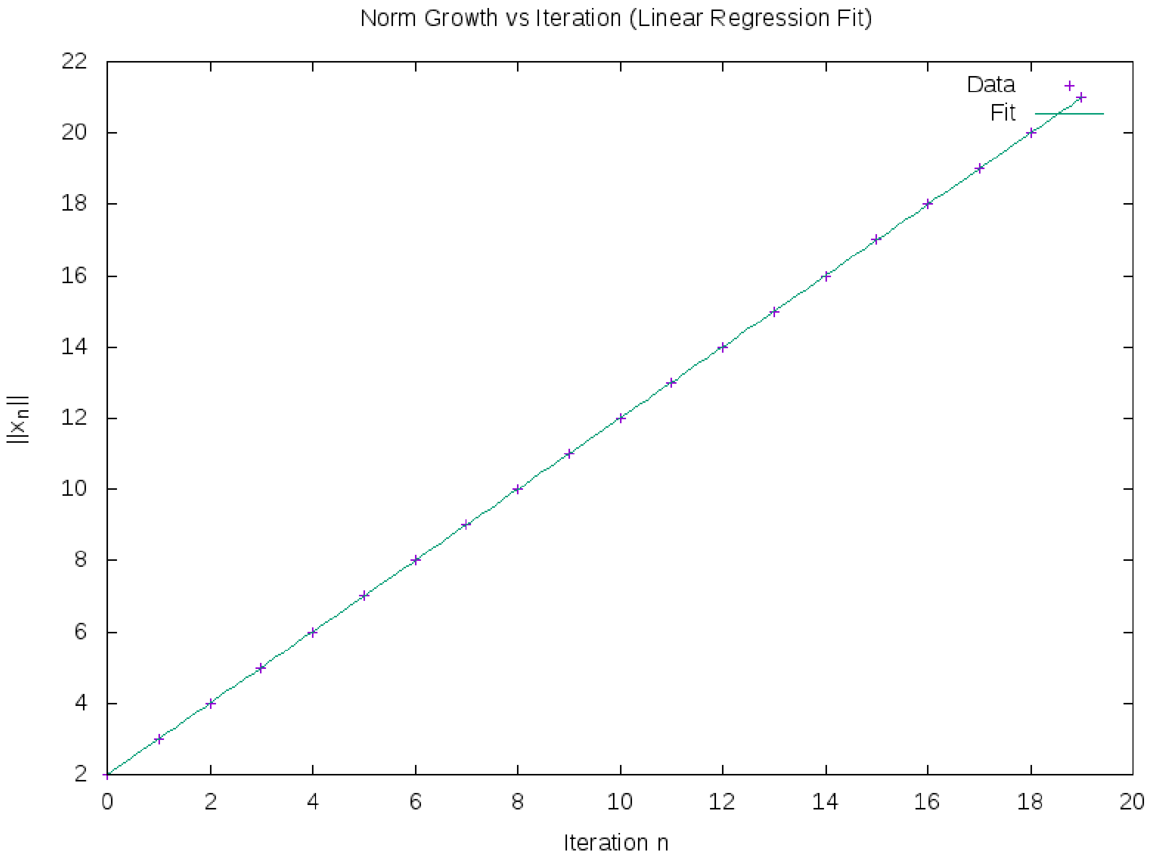

Figure 3.2: Linear Regression and Goodness of Fit Test

The linear regression analysis shows m = 1 and c = 2 for the fitted curve. This gives the result for the deterministic divergence model as y=n+c.

The goodness of fit test, R^2, encoded in our programs was evaluated, and the result shows: $R^2 = 1$. This result shows a perfect fit of the simulated data to the regression line.

The numerical simulations reveal a consistent divergence of trajectories under the iteration. No convergence or bounded behavior is observed, confirming that the operator does not satisfy classical contraction properties (Goebel & Kirk, 1990; Cordero & Vassileva, 2016; Aboukhisheen, 2025).

The most significant result is the emergence of a linear growth law. This behavior is highly stable across iterations and aligns closely with the fitted linear model. The growth is driven primarily by the central coordinate, which accumulates contributions from repeated commutator evaluations.

These findings demonstrate that the iteration produces deterministic, structured divergence rather than chaotic or irregular growth.

The observed linear growth behavior contrasts sharply with classical fixed-point iteration theory, where convergence is the primary objective (Mann, 1953; Chidume, 2009; Ahmad & Admed, 2014; Ahmad & Kamal, 2009). In the present case, the absence of contractivity leads instead to a predictable divergence regime.

This phenomenon can be understood through the algebraic structure of the Heisenberg group. The central component acts as an accumulator of commutator effects, producing a steady increase in magnitude. Such behavior is consistent with known growth properties of nilpotent groups (Stoll, 1996; Goel, 2023).

From a broader perspective, the results highlight a new class of iterative dynamics where algebraic non-commutativity replaces metric contraction as the dominant mechanism. This opens new avenues for extending iterative methods to non-Euclidean and group-based settings (Shumyatsky, 2005).

Future work may incorporate relaxation parameters inspired by Mann-type schemes to explore transitions between divergence and convergence regimes, as well as extensions to higher-dimensional Lie groups and applications in computational physics.

4. Conclusion

This study has presented a computational analysis of a commutator-driven iterative dynamical system on the Heisenberg group. By formulating the iteration as a nonlinear operator induced by group commutation, the work departs from classical fixed-point frameworks and instead explores dynamics governed purely by non-abelian algebraic structure.

The principal outcome of the investigation is the identification of a robust linear growth law in the group norm of the iterates. This behavior arises consistently from the computational experiments and is shown to be intrinsically linked to the accumulation of commutator contributions in the central component of the group.

Unlike traditional iterative schemes that aim for convergence under contractive or nonexpansive conditions (Mann, 1953; Browder, 1965; Goebel & Kirk, 1990), the present system exhibits a deterministic divergence regime driven by non-commutativity.

The findings highlight the significant role of algebraic structure in shaping dynamical behavior. In particular, the Heisenberg group provides a minimal yet nontrivial setting in which commutator interactions produce predictable macroscopic growth, consistent with known structural properties of nilpotent groups (Stoll, 1996; Robinson, 2004). This demonstrates that linear growth can emerge naturally from internal group operations without the need for external forcing or stochastic effects.

From a computational perspective, the simplicity of the implemented scheme underscores its utility as a prototype model for studying nonlinear dynamics on Lie groups. The results suggest that commutator-driven iterations may serve as a foundation for developing new classes of algorithms in non-commutative settings, complementing existing methods in nonlinear analysis and numerical computation (Xu, 2002; Chidume, 2009).

Future work may focus on extending the present framework in several directions, including the introduction of relaxation parameters to recover convergence behavior, the analysis of higher-dimensional or more general Lie groups, and the exploration of applications in computational physics where group-based structures arise naturally.

Overall, this study establishes a clear link between commutator dynamics and linear growth phenomena, opening new avenues for both theoretical and computational investigation.

References

1. Aboukhisheen, K. (2025). The variational iteration method: A unified framework for nonlinear differential equations with modern enhancements. Azzaytuna University Journal, 2025, 56.

2. Ahmad, A. E.-S., & Ahmed, S. A. (2014). Fixed points by certain iterative schemes with applications. Fixed Point Theory and Applications, 2014, 121.

3. Ahmad, A. E.-S., & Kamal, A. (2009). Construction of fixed points by some iterative schemes. Fixed Point Theory and Applications, 2009, 612491.

4. Ahmad, F., Samraiz, M., Ahmad, J., & De la Sen, M. (2026). AK-iterative scheme for fixed point approximation in hyperbolic metric spaces and applications in Volterra integral equation. Fixed Point Theory, Algorithms and Applications, 2026, 7.

5. Akram, S., Junjua, D. U. D., Akram, F. & Khalid, L. (2025). Iterative techniques for nonlinear equations: Addressing multiple roots with unknown multiplicity. In polynomials-Exploring fundamental mathematical expressions. Intech Open.

6. Alharthi, N. H., Okeke, G. A., Udo, A. V., Alqahtani, R. T., & Yusuf, A. (2026). Novel iterative method for approximation of fixed points of generalized (α, β)-nonexpansive mappings with applications to SEIR epidemic model. Scientific Reports, 16, 11833.

7. Baillon, J. B. (1975). Un théorème de type ergodique pour les contractions non linéaires. Comptes Rendus de l'Académie des Sciences de Paris, Série A-B, 280, 1511–1514.

8. Bauschke, H. H., & Combettes, P. L. (2017). Convex analysis and monotone operator theory in Hilbert spaces (2nd ed.). Springer.

9. Browder, F. E. (1965). Nonexpansive nonlinear operators in a Banach space. Proceedings of the National Academy of Sciences of the United States of America, 54(4), 1041–1044.
10. Chidume, C. E. (2009). Geometric properties of Banach spaces and nonlinear iterations. Springer.
11. Cordero, A., Torregrosa, J. R., & Vassileva, M. P. (2016). Design, analysis, and applications of iterative methods for solving nonlinear systems. In D. Lee et al. (Eds.), Nonlinear systems: Design, analysis, estimation and control. IntechOpen.
12. Dehaish, B. A., Khamsi, M. A., & Kozłowski, W. M. (2015). On the convergence of iteration processes for semigroups of nonlinear mappings in modular function spaces. Fixed Point Theory and Applications, 2015, 3.
13. Goebel, K., & Kirk, W. A. (1990). Topics in metric fixed point theory. Cambridge University Press.
14. Goel, S. (2023). Fixed point theory and iterative procedures: From classical theorems to modern developments. Asian Journal of Mathematical Sciences, 7(1).
15. Grau-Sánchez, M., Noguera, M., & Gutiérrez, J. M. (2015). New iterative technique for solving a system of nonlinear equations. Applied Mathematics and Computation, 271, 446–466.
16. Hall, P. (1934). Nilpotent groups. Proceedings of the London Mathematical Society, 38, 29–95.
17. Kirk, W. A., & Sims, B. (Eds.). (2001). Handbook of metric fixed point theory. Springer.
18. Mann, W. R. (1953). Mean value methods in iteration. Proceedings of the National Academy of Sciences, 39(6), 506–510.
19. Mohamadi, I. (2011). Iterative methods for variational inequalities over the intersection of fixed point sets of nonexpansive semigroups in Banach spaces. Fixed Point Theory and Applications, 2011, Article 620284.
20. Okorie, C. E., Johnson, B. O., Michael, A. I., & Fidelis, A. T. (2018). Comparison of some iterative methods of solving nonlinear equations. International Journal of Theoretical and Applied Mathematics, 4(2), 22–28.
21. Robinson, D. J. S. (2004). A course in the theory of groups (2nd ed.). Springer.
22. Saqib, M., Iqbal, M., Ahmed, S., Ali, S., & Ismaeel, T. (2015). New modification of fixed point iterative method for solving nonlinear equations. Applied Mathematics, 6(11), 1857–1863.
23. Saqib, M., Radwan, T., & Alharbi, R. (2026). Iterative methods for nonlinear equations in coupled systems using modified homotopy perturbation. Ain Shams Engineering Journal, 17(2), 103965.
24. Shah, F. A., & Noor, M. A. (2024). Exploring effective iterative methods for nonlinear equations using variational iteration technique. Engineering and Applied Science Letters, 7(1), 1–9.
25. Shumyatsky, P. (2005). Applications of Lie ring methods to group theory. Journal of Algebra, 292(2), 507–524.
26. Stoll, M. (1996). On the asymptotics of growth functions of groups. Journal of the London Mathematical Society, 54(1), 67–80.
27. Wartono, W., & Suryani, I. (2020). The solution of nonlinear parabolic equation using variational iteration method. Jurnal Matematika, Statistika dan Komputasi, 16(3), 287–295.

28. Xu, H.-K. (2002). Iterative algorithms for nonlinear operators. Journal of the London Mathematical Society, 66(1), 240–256.

29. Zhang, L., Wu, Q.-B., Chen, M.-H., & Lin, R.-F. (2021). Two new effective iteration methods for nonlinear systems with complex symmetric Jacobian matrices. Computational and Applied Mathematics, 40, 97.
