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A Numerical Method for Solving First-Order Nonlinear Volterra-Fredholm Integro-Differential Equations Using Power Series Polynomials

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Abstract: This is a numerical approach for solving first-order Nonlinear Volterra-Fredholm Integro-differential equations with power series polynomials. The approach approximated Volterra-Fredholm Integro-differential equations using power series polynomials. The modelled problem is converted to a system of algebraic equations, which is then solved using the standard collocation approach. After determining the approach's uniqueness and convergence, numerical examples were used to assess its effectiveness. The results showed that the method is competitive with other methods. The proposed method transforms the given Integro-differential equation into an equivalent system of algebraic equations by approximating the unknown function with a finite power series expansion. The coefficients of the series are determined by substituting the approximation into the original equation and enforcing the equality at selected collocation points within the domain of interest. This technique ensures high accuracy and rapid convergence with minimal computational effort. Several illustrative examples are provided to demonstrate the efficiency and reliability of the method. The obtained results show excellent agreement with exact solutions, confirming the suitability of the power series polynomial approach for solving first-order Volterra–Fredholm Integro-differential equations arising in applied mathematics, physics and engineering models.

Keywords: Power Series Polynomial, Volterra-Fredholm Integro-Differential Equation, Standard Collocation Method, Nonlinear Integro-Differential Equations, Numerical Approximation, Convergence Analysis.

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INTRODUCTION

In this paper, a computational method based on power series polynomials is used for solving first-order nonlinear Volterra-Fredholm Integro-differential equations. The method constructs an approximate polynomial solution by incorporating the initial condition and evaluating the differential and integral terms within the power series framework. The performance of the proposed method is assessed through selected numerical examples and the results are compared with exact solutions and existing methods to demonstrate its accuracy, convergence and computational efficiency. The numerical results indicate that the proposed approach provides reliable approximations with relatively few polynomial terms, making it a practical and efficient technique for solving this class of nonlinear Integro-differential equations.

Integro-differential equations are widely used in mathematical modeling of real-world phenomena where both local dynamics and historical or spatial interactions are important. In particular, Volterra–Fredholm Integro-differential equations combine Volterra-type memory effects with Fredholm-type global interactions, making them useful in fields such as physics, engineering and biological systems modeling. These equations have attracted increasing attention due to their ability to describe complex nonlinear processes that cannot be captured by classical differential equations alone (Brunner, 2017; Rahman et al., 2021).

First-order nonlinear Volterra–Fredholm integro–differential equations frequently arise in applications involving heat transfer, fluid flow, viscoelastic materials and population dynamics. However, obtaining exact analytical solutions for such equations is often difficult due to the presence of nonlinear terms and mixed integral operators. As a result, the development of efficient and accurate numerical methods remains an active area of research (Khan et al., 2020).

Over the past two decades, several computational techniques have been proposed for solving Integro-differential equations. These include spectral methods, collocation methods, wavelet-based techniques and decomposition approaches. For instance, Chebyshev spectral and wavelet methods have been successfully applied to Volterra–Fredholm problems, demonstrating high accuracy and fast convergence (Zhou & Xu, 2018; Ali et al., 2019). Similarly, least squares and polynomial-based methods have been developed to handle nonlinear Integro-differential equations efficiently (Benzahi *et al*., 2023).

Despite these advances, many existing methods still face challenges such as computational complexity, slow convergence for strongly nonlinear problems as well as difficulty in handling mixed Volterra–Fredholm operators. These limitations have motivated the search for alternative approaches that are both simple and computationally efficient.

A differential transformation approach was used to solve high-order nonlinear Volterra-Fredholm Integro-differential equations using separable kernels. Several cases are taken into consideration, and the outcomes showed that the differential transformation method's process is straightforward and efficient, and that it may yield an exact or precise approximation of the solution (Salah Behiry, Saied & Mohamed, 2012).

Jinjiao *et al*. (2021) presented the nonlinear Volterra-Fredholm Integro-Differential Equations (V-FIDE) using a novel approach that combines the simplified reproducing kernel method (SRKM) and the homotopy perturbation method (HPM). First, nonlinear issues were transformed into linear ones using the HPM. The linear issues are then resolved using the SRKM. Second, it was demonstrated that the estimated solution converges uniformly. Lastly, a few numerical computations are suggested to confirm the method's efficacy.

Asiya and Najmuddin (2023) considered the approximate solutions for the Volterra-Fredholm Integro-differential equations using the Adomian Decomposition Method (ADM) and the Modified Adomian Decomposition Method (MADM). Furthermore, they demonstrated the solutions' existence, uniqueness and convergence. An algorithm was utilized to get to that conclusion. Additionally, the convergence study of the suggested method is carried out using examples of deterministic models in tables and graphs.

Agbolade and Anake (2017) studied numerical solutions of linear Integro-differential equations of the Volterra type. A power series was the fundamental polynomial used to approximate the solution. Furthermore, standard collocation and Chebyshev-Gauss-Lobatto collocation were used, respectively to obtain the approximate solution.

Ahmed (2021) presented a novel approach to solving Volterra integral equations in their research. The Chebyshev spectral collocation method served as the foundation for the procedure. Applying the suggested strategy results in a system of easily solved algebraic equations from the Volterra integral equation.

Ajileye (2023) presented a collocation method for solving multi-order fractional differential equations with initial conditions in the Caputo sense. The integral form of the equation was converted into a system of linear algebraic equations. To obtain the numerical results, the algebraic equations were solved using matrix inversion, and the solutions were then entered into the approximate equation.

In recent years, power series and polynomial approximation methods have gained attention due to their straightforward implementation and strong analytical structure. These methods approximate the solution as a truncated series expansion, converting the Integro-differential equation into a system of algebraic recurrence relations. This approach avoids discretization of the full domain and reduces computational cost while maintaining accuracy (Hussain et al., 2022; Prentice, 2023).

Motivated by these developments, this paper presents a computational method based on power series polynomials for solving first-order nonlinear Volterra–Fredholm Integro-differential equations. The method constructs the solution in a truncated power series form where the coefficients are determined recursively. The proposed technique is simple to implement, avoids linearization and provides an efficient framework for nonlinear integral problems.

Vito Volterra conducted research in the early 1900s and came up with what is known as integral-differential equations. The integral sign writing below and the ordinary derivative together describe the unknown function $y(x)$ in this kind of equation.

$$y(x) = f(x) + \int_0^x k_1(x,t)y(t)dt + \int_0^1 k_2(x,t)y(t)dt \qquad (1.1)$$

The aforementioned formula is known as the Volterra-Fredholm Integro-differential equation (Ajileye *et al,*2022).

Ajileye *et al*. (2022) considered a first-order Nonlinear Volterra-Fredholm Integro-differential equation of the form

$$y^{(\propto)}(x) + g(x)y(x) + \lambda_1 \int_0^x K_1(x,t)y(t)^m dt + \lambda_2 \int_0^1 K_2(x,t)y(t)^m dt = F(x) \qquad (1.2)$$

Subject to the initial condition

$$y(0) = a, m\geq 1, a = 1, \alpha\geq 1 \qquad (1.3)$$

where $K_1(x,t)$ and $K_2(x,t)$ are the Volterra and Fredholm integral kernel function respectively, λ_1, and $g(x)$ are known constants. $F(x)$ is the known function and $y^{(\alpha)}(x)$ is the first derivative of the unknown function $y(x)$ to be determined. If $\lambda_1 = 0$, equation (1.1) becomes the Fredholm Integro-differential equation. If $\lambda_2 = 0$, it becomes the Volterra integro-differential equation.

The main objective of this study is to develop a reliable numerical scheme for approximating solutions of the considered equations and to demonstrate its accuracy and convergence through illustrative examples.

2. MATHEMATICAL FORMULATIONS

Using an approximate solution of the form:

$$y(x) = \sum_{n=0}^{N} a_n x^n \qquad (2.4)$$

where a_n is the coefficients to be determined.

Differentiating equation (1) yields

$$y'(x) = \sum_{n=0}^{N} a_n n x^{n-1} \qquad (2.5)$$

Substituting equation (4) and (5) into equation (2) to get the N-term

$$\sum_{n=0}^{N} a_n n x^{n-1} + g(x)\left(\sum_{n=0}^{N} a_n x^n\right) + \lambda_1 \int_0^x K_1(x,t)\left(\sum_{n=0}^{N} a_n t^n\right)^m dt + \lambda_2 \int_0^1 K_2(x,t)\left(\sum_{n=0}^{N} a_n t^n\right)^m dt = F(x)$$

(2.6)

Collecting the like terms of equation (3) gives

$$\sum_{n=0}^{N} a_n \left[n x^{n-1} + g(x)\left(x^n\right) + \lambda_1 \int_0^x K_1(x,t)\left(t^n\right)^m dt + \lambda_2 \int_0^1 K_2(x,t)\left(t^n\right)^m dt \right] = F(x) \quad (2.7)$$

Hence, Equation (2.7) can be written as

$$\sum_{n=0}^{N} a_n \beta_n(x) = F(x) \quad (2.8)$$

Where

$$\beta_n(x) = n x^{n-1} + g(x)\left(x^n\right) + \lambda_1 \int_0^x K_1(x,t)\left(t^n\right)^m dt + \lambda_2 \int_0^1 K_2(x,t)\left(t^n\right)^m dt$$

Equation (2.8) can be further re-written in the form

$$\beta_n(x) A = F(x) \quad (2.9)$$

where

$$\beta_n(x) = \left[\beta_0(x) \quad \beta_1(x) \cdots \beta_n(x)\right], \quad \mathbf{A} = \left[a_0 a_1 \cdots a_n\right]^T$$

Using the Standard Collocation Method (SCM) taken within an interval [a, b] we have:

$$x_i = \frac{a+(b-a)i}{N} \qquad i = 1,2,3\ldots.N$$

we obtain

$$\beta_n\left(x_i\right) A = F(x_i) \quad (2.10)$$

where

$$\beta_i(x_i) = \begin{bmatrix} \beta_0(x_0) & \beta_1(x_0) & \beta_2(x_0) & \boxed{?} & \beta_N(x_0) \\ \beta_0(x_1) & \beta_1(x_1) & \beta_2(x_1) & \boxed{?} & \beta_N(x_1) \\ \boxed{?} & \boxed{?} & \boxed{?} & \boxed{?} & \boxed{?} \\ \beta_0(x_N) & \beta_1(x_N) & \beta_2(x_N) & \boxed{?} & \beta_N(x_N) \end{bmatrix}, F(x_i) = \begin{bmatrix} F(x_0) \\ F(x_1) \\ \boxed{?} \\ F(x_N) \end{bmatrix}$$

2.1 Consideration of Initial Condition

Using the initial condition given below:

$$y(0) = a \quad (2.11)$$

Adding equation (11) to equation (10) gives

$$\beta_i^*(x_i)A = F^*(x_i) \tag{2.12}$$

Where

$$\beta_i^*(x_i) = \begin{bmatrix} \beta_{00} & \beta_{01} & \beta_{02} & \ldots\ldots & \beta_{0n} \\ \vdots & & & & \\ \beta_{10} & \beta_{11} & \beta_{12} & \ldots\ldots & \beta_{1n} \\ \vdots & & & & \\ \beta_{m0} & \beta_{m1} & \beta_{m2} & \ldots\ldots & \beta_{mn} \\ \vdots & & & & \\ U_{00} & U_{01} & U_{02} & \ldots\ldots & U_{0n} \\ \vdots & & & & \\ U_{m0} & U_{m1} & U_{m2} & \ldots\ldots & U_{mn} \end{bmatrix}$$

$F^*(x_i) = [F_0 F_1 \ldots\ldots\ldots F_{n-m-1} F_{n-m} \ldots\ldots\ldots\ldots\ldots F_N]^T$

Equation (12) is the algebraic equation that is being solved to obtain the values of a_n and substituted into the approximate solution to give the numerical solution.

3 RESULT AND DISCUSSION

Here, the standard collocation method was applied to the solution of the first-order nonlinear Volterra Fredholm Integro-differential equation using a power series polynomial as a basis function. We discuss the result by reducing the first-order integro-differential equation to a system of linear algebraic equations. We equally discussed the result of the uniqueness of the solution.

3.1 Uniqueness of the Method

Theorem 3.1: Banach's fixed point theorem (Zhou, 2014)

Let (X, d) be a complete metric space, then each contraction mapping $T: X\rightarrow X$ has a unique fixed point x of T in X, such that $T(x) = x$

In order to establish the method's uniqueness, we provide the following hypothesis:

H1: Let $T : X \rightarrow X$ be a mapping for y_1; $y_1 \in X$; There exist a constant, $L > 0$: such that

$$|y_1^m(t) - y_2^m(t)| \leq L|y_1(t) - y_2(t)|$$

H2: There exist two functions k_1^* and $k_2^* \in C([0, 1]\times[0, 1], R)$, the set of all positive functions such that

$$k_1^* = max_{x\in[0,1]}|\lambda_1|\int_0^x |K_1(x,t)|dt < \infty$$

$$k_2^* = max_{x\in[0,1]}|\lambda_2|\int_0^1 |K_2(x,t)|dt < \infty$$

H3: The function $g\in R$ is continuous such that

$$q^* = max_{x\in[0,1]}|q(x)|$$

In order to establish the uniqueness of the solution, we first prove that

(i) The mapping T is continuous

(ii) T is a q-contraction

Lemma 3.2: Let T:$X\rightarrow X$ be a mapping defined by Theorem 4.1 for y_1, $y_2\in X$. T is q-contraction if and only if

$$\left(\frac{q^*+k_1^*+k_2^*}{\Gamma(\alpha+1)}\right) < 1$$

Proof

Add ${}_{0}I_x^\alpha$ to both sides of equation (1.2) gives

$$0I_x^\alpha y^{(\alpha)}(x) + oI_x^\alpha g(x)y(x) + oI_x^\alpha\lambda_1\int_0^x K_1(x,t)y(t)^m dt + oI_x^\alpha\lambda_2\int_0^1 K_2(x,t)y(t)^m dt = oI_x^\alpha F(x) \quad (4.1)$$

Applying equation (1.6) to equation (1.2) we have

Substitute for $0I_x^\alpha\left(y^{(\alpha)}(x)\right) = y(x) - \sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k$, and $0I_x^\alpha(z(s)) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}z(s)ds$

$$y(x) - \sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k oI_x^\alpha g(x)y(x) + oI_x^\alpha\lambda_1\int_0^x K_1(x,t)y(t)^m dt + oI_x^\alpha\lambda_2\int_0^1 K_2(x,t)y(t)^m dt = oI_x^\alpha F(x) \quad (4.2)$$

Substitute $0I_x^\alpha = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}F(s)ds$ into equation (4.2)

$$y(x) - \sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}F(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}(g(x)y(x))ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\lambda_1\int_0^x K_1(x,t)y(t)^m dt + \lambda_2\int_0^1 K_2(x,t)y(t)^m dt\right)ds$$
$$= \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}F(x)ds \quad (4.3)$$

Let $\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}F(s)ds$= w($x$) from equation (4.3)

$$y(x) - W(x) - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}(g(x)y(x))ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(\lambda_1 \int_0^x K_1(x,t)y(t)^m dt + \lambda_2 \int_0^1 K_2(x,t)y(t)^m dt\right)ds = \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}F(x) \tag{4.4}$$

Making$y(x)$ the subject of the formula in equation (4.4) we have

$$y(x) = w(x) + \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}F(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}(g(x)y(x))ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(\lambda_1 \int_0^x K_1(x,t)y(t)^m dt + \lambda_2 \int_0^1 K_2(x,t)y(t)^m dt\right)d + s\frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}F(x)ds \tag{4.5}$$

Where F(x) = 0

$$y(x) = W(x) - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}(q(x)y(x))ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(\lambda_1 \int_0^x K_1(x,t)y(t)^m dt + \lambda_2 \int_0^1 K_2(x,t)y(t)^m dt\right)ds \tag{4.6}$$

Let $_1(x)$, $y_2(x)\varepsilon X$, applying Banach fixed point to equation (4.2) gives

$$\left(Ty_1\right)(x) = W(x) - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(q(x)y_1(x)\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(\lambda_1 \int_0^x K_1(x,t)y_1(t)^m dt + \lambda_2 \int_0^1 K_2(x,t)y_1(t)^m dt\right)ds \tag{4.7}$$

and

$$\left(Ty_2\right)(x) = W(x) - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(q(x)y_2(x)\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(\lambda_1 \int_0^x K_1(x,t)y_2(t)^m dt + \lambda_2 \int_0^1 K_2(x,t)y_2(t)^m dt\right)ds \tag{4.8}$$

Subtracting equation (4.5) from equation (4.3) gives

$$\left(Ty_1\right)(x) - \left(Ty_2\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left[q(x)\left(y_1(x) - y_2(x)\right)\right]ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(\lambda_1\int_0^x K_1(x,t)(y_1^m(t) - y_2^m(t))dt + \lambda_2\int_0^1 K_2(x,t)(y_1^m(t) - y_2^m(t))dt\right)ds$$

Taking the absolute value of both sides gives

$$\left|\left(Ty_1\right)(x)-\left(Ty_2\right)(x)\right|=\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left|q(x)\left(y_1(x)-y_2(x)\right)\right|ds+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(|\lambda_1|\int_0^x\left|K_1(x,t)\right|\left|(y_1^m(t)-y_2^m(t))\right|dt+|\lambda_2|\int_0^1\left|k_2(x,t)\right|\left|(y_1^m(t)-y_2^m(t))\right|dt\right)ds$$

Taking the maximum of both sides and applying $H_1 - H_3$ it gives $d(Ty_1, Ty_2)\leq\left(\frac{q^*+k_1^*+k_2^*}{\Gamma(\alpha+1)}\right)d(y_1, y_2)$

By the Banach contraction principle, T is a strict contraction mapping.

Lemma 3.3: (Continuity) Let (X, d) be a metric space and $T: X\rightarrow X$ be a mapping, let $y(x), y(x)\varepsilon X$ and the $y_n(x) \;= y(x)$. T is continuous if $d(Ty_n, Ty)\rightarrow 0\ as\ n\rightarrow\infty$.

Proof

$$(Ty)(x)= W(x)-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}(q(x)y(x))ds-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\lambda_1\int_0^x K_1(x,t)y(t)^m dt+\lambda_2\int_0^1 K_2(x,t)y(t)^m dt\right)ds$$

$$\left|\left(Ty_N\right)(x)-(Ty)(x)\right|=\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left|q(x)\left(y(x)-y_N(x)\right)\right|ds+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(|\lambda_1|\int_0^x\left|K_1(x,t)\right|\left|(y(t)-y_N(t))\right|dt+|\lambda_2|\int_0^1\left|K_2(x,t)\right|\left|(y(t)-y_N(t))\right|dt\right)ds$$

Taking the maximum of both sides and using H_1 to H_3 gives

$$\left|\left(Ty_N\right)(x)-(Ty)(x)\right|\leq\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}|q(x)|\max_{x\in[0.1]}\left|\left(y(x)-y_N(x)\right)\right|ds+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(|\lambda_1|\max_{x\in[0.1]}\int_0^x\left|K_1(x,t)\right|\max_{x\in[0.1]}\left|(y(t)-y_N(t))\right|dt+|\lambda_2|\max_{x\in[0.1]}\int_0^1\left|K_2(x,t)\right|\max_{x\in[0.1]}\left|(y(t)-y_N(t))\right|dt\right)ds$$

$$d(Ty_N(x), Ty(x))\leq\left(\frac{q^*+k_1^*+k_2^*}{\Gamma(\alpha+1)}\right)d(y_N, y)$$

Since $d(y_N, y)\rightarrow 0\ as\ n\rightarrow\infty$.as n→∞, then $d(Ty_N, Ty)\rightarrow 0\ as\ n\rightarrow\infty$., therefore T is continuous on [0,1].

3.4 Convergence of the method

Lemma 3.4: (Convergence of method) Let (X, d) be a metric space and $T: X \to X$ be a continuous mapping and $y_N(x), y_{N-1}(x) \varepsilon X$ are approximate solutions of equation (1.2). Let $\Delta_N(x) = |y_N(x) - y_{N-1}(x)|$, $if\ lim_{N\to 0}(\Delta_N(x)) \to 0$, then the method converges to exact solution.

Proof

Let $y_1(x)$ and $y_2(x)$ be approximated by $y_N(x) = \sum_{n=0}^{M} a_n x^n = \emptyset(x)\, A$ and $y_{N-1}(x) = \sum_{n=0}^{M} a_n x^n = \emptyset(x) B$.

Substituting the approximate solution into equation (4.1) gives

$$Ty_N(x) = W(x) - \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} (q(x)\emptyset(x)A) ds - \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} \left(\lambda_1 \int_0^x K_1(x, t)(\emptyset(x)A) dt + \lambda_2 \int_0^1 K_2(x, t)(\emptyset(x)A) dt \right) ds$$

Similarly

$$Ty_{N-1}(x) = W(x) - \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} (q(x)\emptyset(x)B) ds - \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} \left(\lambda_1 \int_0^x K_1(x, t)(\emptyset(x)B) dt + \lambda_2 \int_0^1 K_2(x, t)(\emptyset(x)B) dt \right) ds$$

$$\left| \left(Ty_N\right)(x) - \left(Ty_{N-1}\right)(x) \right| = \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} q(x)\emptyset(x)|B - A| ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} \left(\lambda_1 \int_0^x K_1(x, t)\emptyset(x)|B - A| dt + \lambda_2 \int_0^1 K_2(x, t)\emptyset(x)|B - A| dt \right) ds$$

Since $x \varepsilon [0, 1]$ and $|B - A| \neq 0$, hence $lim_{N\to 0} \Delta_N(x) \to 0$

Therefore, the method of solution converges.

3.5 Numerical Examples

In this section, we provided a set of numerical illustrations to assess the applicability and accuracy of the method. Let the approximate and exact solutions be $y_n(x)$ and*y(x)* respectively. $Error_N = |y_n(x) - y(x)|$.

Example 1: (Ansari *et al.*, 2023) Considering the first-order non-linear Volterra-Fredholm Integro-differential equation given as;

$$y'(x) + xy(x) - \int_0^x y^2(t)dt - \int_0^{0.9} xy^3(t)dt = 2x + x^3 - \frac{x^5}{5} - \frac{(0.9)^7}{7}x \quad (4.9)$$

subject to initial conditions

$$y(0) = 0, y'(0) = 0$$

Exact solution: $y(x) = x^2$

Solution 1

Comparing with equation (1.2) and equation (1.3),$q(x) = x$, $\lambda_1 = 1, \lambda_1 = 1K_1(x,t) = 1, K_2(x,t) = x$ and $F(x) = 2x + x^3 - \frac{x^5}{5} - \frac{(0.9)^7}{7}x$

To show q-contraction of example 1, we write the integral form of equation (4.5) and apply Lemma 4.1, to get

$$y(x) = W(x) - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}(xy(x))ds -$$

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(\int_0^x y(t)^2dt + \int_0^1 xy(t)^3dt\right)ds \quad (4.10)$$

where

$$W(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(2x + x^3 - \frac{x^5}{5} - \frac{(0.9)^7}{7}x\right)ds$$

Take the maximum of both sides gives

$$\left|y_1(x) - y_2(x)\right| \leq \; -\frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(x\left|y_1(x) - y_2(x)\right|\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}$$

$$\left(\int_0^x \left|y_1^2(t) - y_2^2(t)\right|dt + \int_0^1 x\left|y_1^3(t) - y_2^3(t)\right|dt\right)$$

$$d\left(Ty_1(x), Ty_2(x)\right) \leq \; -\frac{1}{\Gamma(1)}\int_0^x (x - s)^{1-1}(x)dsL_1d(y_1, y_2) - \frac{1}{\Gamma(1)}\int_0^x (x - s)^{1-1}\left(\int_0^x dt + \int_0^1 xdt\right)dsL_2d(y_1, y_2)$$

Let L=L_1,L_2 and simplifying gives

$$d\left(Ty_1(x), Ty_2(x)\right) \leq -3x^2 L d(y_1, y_2)$$

Therefore, q-contraction is satisfied in $0 < L < 1$

Solving example 1 numerically at N=5 gives

$$y_5 = -1.110223024625\times10^{-15} + 7.105427357601\times10^{-15}x + 0.999999999999x^2 + 1.818989403546 * 10^{-12}x^5$$

Table 4.1: Exact, approximate and absolute error values for example 1

X	Exact	Our method $_{N=5}$	Error$_5$	(Ansari *et al.*, 2023) Error$_5$
0.2	0.040000000000	0.040000000000	0.00	3.40e-3
0.4	0.160000000000	0.160000000000	0.00	1.11e-2
0.6	0.360000000000	0.360000000000	0.00	1.11e-2
0.8	0.640000000000	0.640000000000	0.00	1.24e-2
1.0	1.000000000000	1.000000000000	0.00	4.52e-2

Example 2: (Hou *et al.*, 2021) Considering the first-order non-linear Volterra Integro-differential equation

$$y'(x) + y(x) - 2\int_0^x sin(x)y^2(t)dt = cos(x) + (1 - x)sin(x) + cos(x)sin^2(x) \quad (4.11)$$

subject to initial conditions

$$y(0) = 0$$

Exact solution: $y(x) = sin(x)$

Solution 2

Comparing with equation (1.2) and equation (1.3), $q(x) = 1$,
$\lambda_1 = 2, \lambda_2 = 0 K_1(x,t) = sin(x)$, and $F(x) = cos(x) + (1 - x)sin(x) + cos(x)sin^2(x)$

To show q-contraction of example 2, we write the integral form of equation (4.5) and apply Lemma 4.1 to get

$$y(x) = W(x) - \frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}(y(x))ds +$$

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x - s)^{\alpha-1}\left(2\int_0^x sin(x)y(t)^2 dt\right)ds \quad (4.12)$$

Where

$$W(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha - 1} (cos(x) + (1 - x)sin(x) + cos(x)sin^2(x))ds$$

Take the maximum of both sides gives

$$\left|y_1(x) - y_2(x)\right| \leq - \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha - 1} \left(\left|y_1(x) - y_2(x)\right|\right) ds +$$

$$\frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha - 1} \left(2 \int_0^x sin(x) \left|y_1^2(t) - y_2^2(t)\right| dt\right) ds$$

$$d\left(Ty_1(x), Ty_2(x)\right) \leq - \frac{1}{\Gamma(1)} \int_0^x (x - s)^{1-1} (1) ds L^1 d(y_1, y_2) + \frac{1}{\Gamma(1)} \int_0^x (x - s)^{1-1} \left(2 \int_0^x sin(x) dt\right) ds L_2 d(y_1, y_2)$$

Let $L = L_1, L_2$ and simplifying gives

$$d\left(Ty_1(x), Ty_2(x)\right) \leq - x + x^2 sin(x) L d(y_1, y_2)$$

Therefore, q-contraction is satisfied in $0 < L < 1$

Solving example 2 numerically at N=6 gives

$y_6 =- 4.737052972000\times10^{-10} + 1.000001048642x - 0.1394239556e - 2x^2 - 0.155098519841x^3 - 0.36024075147e - 1x^4 + 0.54202200699e - 1x^5 - 0.10605020725e - 1x^6$

Table 4.2: Exact and approximate values for example 2

X	Exact	Our method$_{N=6}$	Error$_6$	(Hou *et al.*, 2021)Error$_{12}$
0.2	0.198669330800	0.198662679000	6.651800e-6	8.53966e-5
0.4	0.389418342300	0.389440411500	2.20692e-5	1.79180e-4
0.6	0.564642473400	0.564648677600	4.062042e-5	2.5968e-4
0.8	0.717356090900	0.717323556600	2.5674657-5	3.71839e-4
1.0	0.841470984800	0.841481394400	9.6104096e-5	5.01073e-4

Example 3: (Hou *et al.*, 2021) Considering the first-order non-linear Volterra Integro-differential equation

$$y'(x) + \int_0^x (y^2(t) - 2)dt = \frac{1}{5} x^5 \quad (4.13)$$

subject to initial conditions

$$y(0) = 0$$

Exact solution: $y(x) = x^2$

Solution 3

The approximate solution of equation (4.13) at N=4 gives

$y_4 =- 1.110223025000\times10^{-16}x + 1.000000000000x^2 - 3.552713679000\times10^{-15}x^4$

Table 4.3: Exact and approximate values for example 3

X	Exact	Our method$_{N=4}$	Error$_4$	(Hou *et al.*, 2021)Error$_6$
0.2	0.040000000000	0.040000000000	0.00	4.6875e-17
0.4	0.160000000000	0.160000000000	0.00	2.77556e-17
0.6	0.360000000000	0.360000000000	0.00	1.11022e-16
0.8	0.640000000000	0.640000000000	0.00	2.22045e-16
1.0	1.000000000000	1.000000000000	0.00	1.11022e-15

Example 4: (Maadadi *et al.*, 2018) Considering a nonlinear Fredholm integro-differential equation

$$y'(x) - \int_0^1 xy(t)^2 dt = 1 - \frac{x}{3} \quad (4.14)$$

subject to initial conditions

$y(0) = 0$

Exact solution: $y(x) = x$

Solution 4

The approximate solution of equation (4.14) at $N = 3$ gives

$y_3 = 2.720046457648\times10^{-17} + 1.000000000000x$

Table 4.4: Exact and approximate values for example 4

X	Exact	Our method$_{N=3}$	Error$_4$	(Maadadi *et al.*, 2018)Error$_3$
0.2	0.200000000000	0.200000000000	0.00	5.86e
0.4	0.400000000000	0.400000000000	0.00	2.34e
0.6	0.600000000000	0.600000000000	0.00	5.27e
0.8	0.800000000000	0.800000000000	0.00	9.38e

1.0	1.000000000000	1.000000000000	0.00	1.47e

4.1 RESULTS AND DISCUSSION

This section discusses the numerical results that were derived from the solved examples using the derived numerical approach.

The result obtained for Example 1, as shown in Table 1, is that the approximate solution at $N = 5$ gives,

$y_5 = -1.110223024625\times10^{-15} + 7.105427357601\times10^{-15}x + 0.999999999999x^2 + 1.818989403546 * 10^{-12}x^5$

The numerical result converged to an exact solution, and this confirmed that our method performed better than the method proposed by Ansari *et al*. (2023).

In numerical Example 2, as shown in Table 2, the approximate solution at $N = 6$ gives,

$y_6 = -4.737052972000\times10^{-10} + 1.000001048642x - 0.1394239556e - 2x^2 - 0.155098519841x^3 - 0.36024075147e - 1x^4 + 0.54202200699e - 1x^5 - 0.10605020725e - 1x^6$

. The numerical results give a better result compared to the result obtained by (Hou *et al*., 2021) at $N = 12$.

The approximate solution obtained in Example 3 at $N = 4$ gives $y_4 = -1.110223025000\times10^{-16}x + 1.000000000000x^2 - 3.552713679000\times10^{-15}x^4$. The numerical result converged to an exact solution, and this confirmed that our method performed better than the method proposed by (Hou *et al*., 2021), as shown in Table 3.

The approximate solution obtained in Example 4 at N = 3 gives $y_3 = 2.720046457648\times10^{-17} + 1.000000000000x$. The numerical result converged to an exact solution, and this confirmed that our method performed better than the method proposed by (Maadadi *et al*., 2018). as shown in Table 4

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