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Moving Load Induced Dynamic Response of Orthotropic Rectangular Plates with Asymmetric Elastic Edge Restraints on Pasternak Foundations

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Abstract: This study develops closed-form time-domain solutions for the dynamic response of elastically restrained orthotropic rectangular plates resting on Pasternak foundations and subjected to moving distributed loads and moving distributed masses. The formulation accounts simultaneously for orthotropy, elastic edge restraints, foundation stiffness, and rotatory inertia, thereby providing a unified framework for investigating load–structure interaction in supported plate systems. Starting from the coupled governing fourth-order partial differential equations, a generalized Galerkin procedure is employed to reduce the continuous system to a set of coupled second-order ordinary differential equations. A modified Struble asymptotic technique, combined with Laplace-transform analysis, is subsequently used to derive explicit time-domain response expressions. The resulting solutions provide direct insight into modal contributions and resonance conditions. Parametric analyses demonstrate that the rotatory-inertia correction, Winkler modulus, and Pasternak shear modulus can substantially modify the amplitude, frequency characteristics, and resonance behavior of the plate. A central finding is that representing the transported body as a moving distributed mass, with its associated inertia and inertial coupling retained, can produce response characteristics markedly different from those predicted by the conventional moving-force idealization. The proposed closed-form solutions, therefore, provide an efficient analytical tool for parametric assessment of moving-load plate dynamics.

Keywords: Orthotropic Plate, Moving Distributed Mass, Pasternak Foundation, Struble Asymptotics, Laplace Transform.

I. INTRODUCTION

The dynamic response of structures to moving loads is a central problem in applied mechanics with wide relevance to mechanical, structural, and civil engineering, transportation systems, and applied physics. Canonical examples such as railways and bridges illustrate the practical importance of load–structure interaction and have motivated extensive research (Gbadeyan & Agarana, 2014). Numerous extensions of classical plate theories and numerical schemes have been proposed to characterize plate response under moving loads; for orthotropic plates, Ritz-based formulations effectively capture anisotropic stiffness distributions (Dileep, 2023). Mindlin-plate studies likewise advanced understanding of moving-load dynamics, although some early investigations neglected rotary inertia and shear-deformation effects, limiting their applicability to moderately thick plates (Gbadeyan & Dada, 2006). Recent work has emphasized how foundation stiffness and support conditions alter orthotropic-plate dynamics. Adeoye et al. (2024) examined orthotropic plates on bi-parametric foundations, and Adeoye (2024) analyzed amplitude variations in elastically supported orthotropic plates under moving loads.

Numerical strip and finite-strip methods remain important tools for moving-load problems. Using the finite-strip approach, Huang & Thambiratnam (2001) showed that isotropic rectangular plates on Winkler foundations display steady-state responses for constant-speed or zero-initial-velocity loads, while sudden load application produces pronounced transients. More recent Mindlin-plate elastodynamic studies and investigations of spatially distributed loading highlight the response sensitivity to load distribution and foundation interaction (Amiri et al., 2013). Closed-form and semi-analytical approaches like eigenfunction expansions and integral transforms continue to complement numerical work by producing convergent, physically interpretable solutions. Moving-mass problems are often treated via semi-analytical procedures. For thin orthotropic plates, analytical solutions validated against finite-element models clarify how mode shapes influence dynamic response (Deghboudj et al., 2020), and finite-integral-transform techniques have produced solutions for mixed boundary conditions (Quian et al., 2020).

A notable limitation of many analytic techniques is their reliance on concentrated-load models and idealized boundary conditions. Concentrated masses are rare in practice: realistic moving loads are better represented as distributed masses or area-distributed contact pressures, and this distinction substantially affects inertia coupling, local bending, and foundation interaction (Andi et al., 2014). Consequently, analytical tractability and practical realism remain partially disconnected; explicit treatments of two-dimensional structural members subjected to moving distributed masses are still limited, producing an epistemic gap between closed-form theory and realistic engineering scenarios.

Classical solution strategies provide valuable methodological heritage. Way (1938) introduced polynomial displacement assumptions within an energy framework for clamped

plates; Levy (1942) and Yamak (1961) used double Fourier series to obtain convergent solutions for various boundary conditions (with known limitations at high load magnitudes). Ghosh (1977) studied large-deflection behaviour for rectangular plates on Pasternak foundations, and Berger's linearized formulation has been widely used in early large-deflection studies on elastic supports. Subsequent semi-analytical and numerical extensions like finite-strip and finite-element methods, Galerkin-based vibration studies, and reports of amplitude-dependent shifts in mode shapes and natural frequencies have expanded the toolkit for moving-load analysis (Huang & Thambiratnam, 2001; Silveira et al., 2014; Benamar et al., 1993). Amiri et al. (2013) emphasized again how load distribution and foundation interaction shape Mindlin-type elastodynamic responses.

Recent efforts have begun to bridge analytic and practical models. Adeoye & Awodola (2024) combined variable separable, Struble asymptotics, and Laplace transforms to produce closed-form-like expressions for orthotropic rectangular plates on constant Pasternak foundations. Related studies by Adeoye and co-authors have addressed orthotropic plates under distributed moving masses and bi-parametric foundations using variable separation, asymptotic reduction, and transform techniques to reduce PDEs to tractable ODE systems (Adeoye & Awodola, 2020; Adeoye et al., 2020; Adeoye & Adeoye, 2024). Nevertheless, explicit treatments of orthotropic rectangular plates with free-end boundary conditions under moving distributed loads remain scarce; an important omission, since free edges alter modal characteristics, local bending, and inertia coupling compared with clamped or simply supported edges.

This study addresses that gap by formulating and analyzing the dynamic response of an orthotropic rectangular plate with free ends under moving distributed forces and moving distributed masses. Combining analytical reduction via separation of variables with semi-analytical solution procedures, we systematically explore how plate and foundation parameters shape transient and steady-state responses. Parametric response curves quantify the distinct effects of moving forces versus moving masses and the influence of foundation stiffness, anisotropic flexural rigidities, and boundary freedom on modal coupling and local bending.



Figure 1: A Section of a Plate Bridge

II. MATERIALS AND METHODS

The dynamic response $W(x, y, t)$ of an elastically restrained orthotropic rectangular plate, with constant velocity v_p along a straight line parallel to the x -axis issuing from point $y = \epsilon$ on the y -axis with flexural rigidities D_x and D_y , resting on a constant Pasternak foundation and subjected to moving distributed masses M_p , is governed by the fourth-order partial differential equation:

$$\begin{aligned} D_x \frac{\partial^4}{\partial x^4} W(\mathcal{X}, \mathcal{Y}, t) + 2\sigma \frac{\partial^4}{\partial x^2 \partial y^2} W(\mathcal{X}, \mathcal{Y}, t) + D_y \frac{\partial^4}{\partial y^4} W(\mathcal{X}, \mathcal{Y}, t) + \varphi \frac{\partial^2}{\partial t^2} W(\mathcal{X}, \mathcal{Y}, t) \\ \vartheta h R_0 \left[\frac{\partial^4}{\partial x^2 \partial t^2} W(\mathcal{X}, \mathcal{Y}, t) + \frac{\partial^4}{\partial y^2 \partial t^2} W(\mathcal{X}, \mathcal{Y}, t) \right] + K_0 W(\mathcal{X}, \mathcal{Y}, t) - G_0 \left[\frac{\partial^2}{\partial x^2} W(\mathcal{X}, \mathcal{Y}, t) \right. \\ \left. + \frac{\partial^2}{\partial y^2} W(\mathcal{X}, \mathcal{Y}, t) \right] - \sum_{p=1}^N [M_p g H(x - v_p t) H(y - \epsilon) - M_p \left(\frac{\partial^2}{\partial t^2} W(\mathcal{X}, \mathcal{Y}, t) + 2v_p \right. \\ \left. \frac{\partial^2}{\partial x \partial t} W(\mathcal{X}, \mathcal{Y}, t) + v_p^2 \frac{\partial^2}{\partial x^2} W(\mathcal{X}, \mathcal{Y}, t) \right) H(x - v_p t) H(y - \epsilon) W(\mathcal{X}, \mathcal{Y}, t)] = 0 \end{aligned} \quad (1)$$

where D_x and D_y are the flexural rigidities of the plate along x and y axes respectively as defined below

$$D_x = \frac{\epsilon_x h^3}{12(1-\nu_x \nu_y)}, \quad D_y = \frac{\epsilon_y h^3}{12(1-\nu_x \nu_y)}, \quad \sigma = D_x D_y + \frac{G_o h^3}{6}$$

ϵ_x and ϵ_y are the Young's moduli in the X - and Y -directions, G_o is the Pasternak shear modulus, ν_x and ν_y are the Poisson's ratios satisfying the orthotropic reciprocity relation. Further, ϑ is the plate mass density, h is the plate thickness, t is the time, X and Y the spatial coordinates in the x - and y -directions, R_o the rotary-inertia correction factor, K_o the foundation modulus, g is the acceleration due to gravity, and $H(\cdot)$ denotes the Heaviside function.

Re-writing equation (1), one obtains

$$\begin{aligned} \frac{\partial^2}{\partial t^2} W(\mathcal{X}, \mathcal{Y}, t) + \theta_p^2 W(\mathcal{X}, \mathcal{Y}, t) = R_0 \left[\frac{\partial^4}{\partial x^2 \partial t^2} W(\mathcal{X}, \mathcal{Y}, t) + \frac{\partial^4}{\partial y^2 \partial t^2} W(\mathcal{X}, \mathcal{Y}, t) \right] - \frac{2\sigma}{\varphi} \frac{\partial^4}{\partial x^2 \partial y^2} \\ W(\mathcal{X}, \mathcal{Y}, t) - \frac{D_x}{\varphi} \frac{\partial^4}{\partial x^4} W(\mathcal{X}, \mathcal{Y}, t) - \frac{D_y}{\varphi} \frac{\partial^4}{\partial y^4} W(\mathcal{X}, \mathcal{Y}, t) + [\theta_p^2 - \frac{K_0}{\varphi}] W(\mathcal{X}, \mathcal{Y}, t) + \frac{G_0}{\varphi} \left[\frac{\partial^2}{\partial x^2} \right. \\ \left. W(\mathcal{X}, \mathcal{Y}, t) + \frac{\partial^2}{\partial y^2} W(\mathcal{X}, \mathcal{Y}, t) \right] + \sum_{p=1}^N \left[\frac{M_p g}{\varphi} H(x - v_p t) H(y - \epsilon) - \frac{M_p}{\varphi} \left(\frac{\partial^2}{\partial t^2} W(\mathcal{X}, \mathcal{Y}, t) \right. \right. \\ \left. \left. + 2v_p \frac{\partial^2}{\partial x \partial t} W(\mathcal{X}, \mathcal{Y}, t) + v_p^2 \frac{\partial^2}{\partial x^2} W(\mathcal{X}, \mathcal{Y}, t) \right) H(x - v_p t) H(y - \epsilon) \right] \end{aligned} \quad (2)$$

where θ_p^2 represent natural frequencies, $1 \leq p < \infty$

Without loss of generality, the initial conditions may be written as:

$$W(x, y, t) = 0 = \frac{\partial}{\partial t} W(x, y, t) \quad (3)$$

Using the technique of Shadnam et al. (2001), the transverse deflection is expressed as the double series, which reduces the PDE to a set of ODEs for the modal coefficients.

$$W(x, y, t) = \sum_{p=1}^N \vartheta_p(x, y) \varpi_p(t) \quad (4)$$

where $\vartheta_p(x, y) = \vartheta_p(x) \vartheta_p(y)$ and

$$\begin{aligned} \vartheta_p(x) &= \sin \frac{\varsigma_p}{L_x} x + A_p \cos \frac{\varsigma_p}{L_x} x + B_p \sinh \frac{\varsigma_p}{L_x} x + C_p \cosh \frac{\varsigma_p}{L_x} x \\ \vartheta_p(y) &= \sin \frac{\varsigma_p}{L_y} y + A_p \cos \frac{\varsigma_p}{L_y} y + B_p \sinh \frac{\varsigma_p}{L_y} y + C_p \cosh \frac{\varsigma_p}{L_y} y \end{aligned} \quad (5)$$

The right-hand side of equation (2) may be expanded in the same series of spatial basis functions as

$$\begin{aligned} \sum_{p=1}^{\infty} R_0 \left[\frac{\partial^4}{\partial x^2 \partial t^2} W(\mathcal{X}, \mathcal{Y}, t) + \frac{\partial^4}{\partial y^2 \partial t^2} W(\mathcal{X}, \mathcal{Y}, t) \right] - \frac{2\sigma}{\phi} \frac{\partial^4}{\partial x^2 \partial y^2} W(\mathcal{X}, \mathcal{Y}, t) - \frac{D_x}{\phi} \frac{\partial^4}{\partial x^4} W(\mathcal{X}, \mathcal{Y}, t) \\ - \frac{D_y}{\phi} \frac{\partial^4}{\partial y^4} W(\mathcal{X}, \mathcal{Y}, t) + [\theta_p^2 - \frac{K_0}{\phi}] W(\mathcal{X}, \mathcal{Y}, t) + \frac{G_0}{\phi} \left[\frac{\partial^2}{\partial x^2} W(\mathcal{X}, \mathcal{Y}, t) + \frac{\partial^2}{\partial y^2} W(\mathcal{X}, \mathcal{Y}, t) \right] + \\ \sum_{p=1}^N \left[\frac{M_{pg}}{\phi} H(\mathcal{X} - v_p t) H(\mathcal{Y} - \epsilon) - \frac{M_p}{\phi} \left(\frac{\partial^2}{\partial t^2} W(\mathcal{X}, \mathcal{Y}, t) + 2v_p \frac{\partial^2}{\partial x \partial t} W(\mathcal{X}, \mathcal{Y}, t) + v_p^2 \frac{\partial^2}{\partial x^2} \right. \right. \\ \left. \left. W(\mathcal{X}, \mathcal{Y}, t) \right) H(\mathcal{X} - v_p t) H(\mathcal{Y} - \epsilon) \right] = \sum_{p=1}^N \vartheta_p(\mathcal{X}, \mathcal{Y}) \phi_p(t) \end{aligned} \quad (6)$$

Multiplying equation (6) by $\vartheta_s(X, Y)$ and integrating over the plate area, A and using the orthogonality condition, yields the modal ordinary differential equation:

$$\begin{aligned} \phi_p(t) = \frac{1}{\psi^*} \sum_{a=1}^{\infty} \int_A \left[R_0 \left(\frac{\partial^4}{\partial x^2 \partial t^2} W(\mathcal{X}, \mathcal{Y}, t) + \frac{\partial^4}{\partial y^2 \partial t^2} W(\mathcal{X}, \mathcal{Y}, t) \right) - \frac{2\sigma}{\phi} \frac{\partial^4}{\partial x^2 \partial y^2} W(\mathcal{X}, \mathcal{Y}, t) - \right. \\ \left. \frac{D_x}{\phi} \frac{\partial^4}{\partial x^4} W(\mathcal{X}, \mathcal{Y}, t) - \frac{D_y}{\phi} \frac{\partial^4}{\partial y^4} W(\mathcal{X}, \mathcal{Y}, t) + (\theta_p^2 - \frac{K_0}{\phi}) W(\mathcal{X}, \mathcal{Y}, t) + \frac{G_0}{\phi} \left(\frac{\partial^2}{\partial x^2} W(\mathcal{X}, \mathcal{Y}, t) \right. \right. \\ \left. \left. + \frac{\partial^2}{\partial y^2} W(\mathcal{X}, \mathcal{Y}, t) \right) + \sum_{p=1}^N \left[\frac{M_{pg}}{\phi} H(x - v_p t) H(y - \epsilon) - \frac{M_p}{\phi} \left(\frac{\partial^2}{\partial t^2} W(\mathcal{X}, \mathcal{Y}, t) + 2v_p \frac{\partial^2}{\partial x \partial t} \right. \right. \right. \\ \left. \left. \left. W(\mathcal{X}, \mathcal{Y}, t) + v_p^2 \frac{\partial^2}{\partial x^2} W(\mathcal{X}, \mathcal{Y}, t) \right) H(\mathcal{X} - v_p t) H(\mathcal{Y} - \epsilon) \right] \vartheta_s(\mathcal{X}, \mathcal{Y}) dA \end{aligned} \quad (7)$$

and otherwise zero when $p \neq s$

where

$$\psi^* = \int_A \vartheta_p^2(x, y) dA \quad (8)$$

Using (6) and (7) in (2) and simplifying, we obtain:

$$\begin{aligned} \ddot{\varpi}_p(t) + \theta_p^2 \varpi_p(t) = \frac{1}{\psi^*} \sum_{a=1}^{\infty} \int_A [R_0(R_0(\frac{\partial^2 \vartheta_k(\mathcal{X}, \mathcal{Y})}{\partial \mathcal{X}^2} \vartheta_s(\mathcal{X}, \mathcal{Y}) \ddot{\varpi}_a(t) + \frac{\partial^2(\mathcal{X}, \mathcal{Y})}{\partial \mathcal{Y}^2} \vartheta_s(\mathcal{X}, \mathcal{Y}) \ddot{\varpi}_a(t) \\) - \frac{2\sigma}{\varphi} \frac{\partial^2 \vartheta_a(\mathcal{X}, \mathcal{Y})}{\partial \mathcal{X}^2 \partial \mathcal{Y}^2} \vartheta_s(\mathcal{X}, \mathcal{Y}) \varpi_a(t) - \frac{D_x}{\varphi} \frac{\partial^4 \vartheta_a(\mathcal{X}, \mathcal{Y})}{\partial \mathcal{X}^4} \vartheta_s(\mathcal{X}, \mathcal{Y}) \varpi_a(t) - \frac{D_y}{\varphi} \frac{\partial^4 \vartheta_a(\mathcal{X}, \mathcal{Y})}{\partial \mathcal{Y}^4} \vartheta_s(\mathcal{X}, \mathcal{Y}) \\ \frac{\partial^4 \vartheta_a(\mathcal{X}, \mathcal{Y})}{\partial \mathcal{Y}^4} \vartheta_s(\mathcal{X}, \mathcal{Y}) \varpi_a(t) + (\theta_a^2 - \frac{K_0}{\varphi}) \vartheta_a(\mathcal{X}, \mathcal{Y}) \vartheta_s(\mathcal{X}, \mathcal{Y}) \varpi_a(t) + \frac{G_0}{\varphi} (\frac{\partial^2 \vartheta_a(\mathcal{X}, \mathcal{Y})}{\partial \mathcal{X}^2} \vartheta_s(\mathcal{X}, \mathcal{Y}) \\ \varpi_a(t)) + \sum_{p=1}^N (\frac{M_{pg}}{\varphi} \vartheta_s(\mathcal{X}, \mathcal{Y}) H(\mathcal{X} - v_p t) H(y - \epsilon) - \frac{M_p}{\varphi} (\vartheta_p(\mathcal{X}, \mathcal{Y}) \vartheta_s(\mathcal{X}, \mathcal{Y}) \end{aligned}$$

$$\ddot{\varpi}_a(t) + 2v_p \frac{\partial \vartheta_k(x, y)}{\partial X} \vartheta_s(x, y) \dot{\varpi}_a(t) + v_p^2 \frac{\partial^2 \vartheta_k(x, y)}{\partial X^2} \vartheta_s(x, y) \varpi_a(t) H(x - v_p t) H(y - \epsilon)] dA \quad (9)$$

The preceding equation is a set of coupled, second-order ordinary differential equations. Representing the Heaviside function on the interval $0 < x < 1$ and $0 < y < 1$ by a Fourier series, we write

$$H(x - v_p t) = \frac{1}{4} + \frac{1}{\pi} \sum_{r=1}^{\infty} \frac{\sin(2r+1)\pi(x-v_p t)}{2r+1}, 0 < x < 1 \quad (10)$$

$$H(y - \epsilon) = \frac{1}{4} + \frac{1}{\pi} \sum_{f=1}^{\infty} \frac{\sin(f+1)\pi(y-\epsilon)}{2f+1}, 0 < y < 1 \quad (11)$$

Using (10) and (11) in (9) and simplifying, we obtain

$$\begin{aligned}
 & \ddot{w}_p(t) + \theta_p^2 w_p(t) - \frac{1}{\psi^*} \sum_{a=1}^{\infty} [R_0 \eta_0 \ddot{w}_a(t) - \frac{2\sigma}{\phi} \eta_1 w_a(t) - \frac{D_x}{\phi} \eta_2 w_a(t) - \frac{D_y}{\phi} \eta_3 w_a(t) \\
 & + (\theta_a^2 - \frac{K_0}{\phi}) \eta_4 w_a(t) + \frac{G_0}{\phi} \eta_5 w_a(t) - \sum_{j=1}^N \frac{M_p}{\phi} ((\eta_6 + \frac{1}{\pi^2} (\sum_{r=1}^{\infty} \rho_1^* \frac{\cos(2r+1)\pi v_p t}{2r+1} \\
 & - \sum_{r=1}^{\infty} \rho_2^* \frac{\sin(2r+1)\pi v_p t}{2r+1}) (\sum_{f=1}^{\infty} \rho_3^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_4^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi} \\
 & (\sum_{r=1}^{\infty} \rho_5^* \frac{\cos(2i+1)\pi v_p t}{2r+1} - \sum_{r=1}^{\infty} \rho_6^* \frac{\sin(2r+1)\pi v_p t}{2r+1}) + \frac{1}{4\pi} (\sum_{f=1}^{\infty} \rho_7^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \\
 & \sum_{f=1}^{\infty} \rho_8^* \frac{\sin(2f+1)\pi \epsilon}{2f+1})) \ddot{w}_a(t) + 2v_p t (\eta_7 + \frac{1}{\pi^2} (\sum_{r=1}^{\infty} \rho_9^* \frac{\cos(2r+1)\pi v_p t}{2r+1} - \sum_{b=1}^{\infty} \rho_{10}^* \\
 & \frac{\sin(2r+1)\pi v_p t}{2r+1}) (\sum_{f=1}^{\infty} \rho_{11}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{12}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi} (\sum_{r=1}^{\infty} \rho_{13}^* \\
 & \frac{\cos(2r+1)\pi v_p t}{2r+1} - \sum_{r=1}^{\infty} \rho_{14}^* \frac{\sin(2r+1)\pi v_p t}{2r+1}) + \frac{1}{4\pi} (\sum_{f=1}^{\infty} \rho_{15}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{16}^* \\
 & \frac{\sin(2f+1)\pi \epsilon}{2f+1})) \ddot{w}_a(t) + v_j^2 (\eta_8 + \frac{1}{\pi^2} (\sum_{r=1}^{\infty} \rho_{17}^* \frac{\cos(2r+1)\pi v_p t}{2i+1} - \sum_{r=1}^{\infty} \rho_{18}^* \frac{\sin(2r+1)\pi v_p t}{2r+1} \\
 &) (\sum_{f=1}^{\infty} \rho_{19}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{20}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi} (\sum_{r=1}^{\infty} \rho_{21}^* \frac{\cos(2r+1)\pi v_p t}{2r+1} \\
 & - \sum_{r=1}^{\infty} \rho_{22}^* \frac{\sin(2r+1)\pi v_p t}{2r+1}) + \frac{1}{4\pi} (\sum_{f=1}^{\infty} \rho_{23}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{24}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1})) \\
 & w_a(t))] = \sum_{a=1}^{\infty} \sum_{p=1}^N \frac{M_{pg}}{\phi \eta^*} \vartheta_a(v_p t) \vartheta_s(\epsilon)
 \end{aligned} \quad (14)$$

is the refined equation governing the dynamic behavior of an orthotropic rectangular plate on a constant bi-parametric elastic foundation, subjected to moving distributed load.

where $\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \rho_1^*, \rho_2^*, \rho_3^*, \rho_4^*, \rho_5^*, \rho_6^*, \rho_7^*, \rho_8^*, \rho_9^*, \rho_{10}^*, \rho_{11}^*, \rho_{12}^*, \rho_{13}^*, \rho_{14}^*, \rho_{15}^*, \rho_{16}^*, \rho_{17}^*, \rho_{18}^*, \rho_{19}^*, \rho_{20}^*, \rho_{21}^*, \rho_{22}^*, \rho_{23}^*$ and ρ_{24}^* .

$\vartheta_s(x, y)$ is assumed as the products of functions $\vartheta_s(x) \vartheta_s(y)$ which are the beam functions in the directions of x and y axes. That is

We assume separable spatial modes given by the product of one-dimensional beam functions in the x and y directions, i.e.

$$\vartheta_s(x, y) = \vartheta_s(x) \vartheta_s(y) \quad (15)$$

where

$$\begin{aligned}
 \vartheta_s(\mathcal{X}) &= \sin \delta_s \mathcal{X} + A_s \cos \delta_s \mathcal{X} + B_s \sinh \delta_s \mathcal{X} + C_s \cosh \delta_s \mathcal{X} \\
 \vartheta_s(\mathcal{Y}) &= \sin \delta_s \mathcal{Y} + A_s \cos \delta_s \mathcal{Y} + B_s \sinh \delta_s \mathcal{Y} + C_s \cosh \delta_s \mathcal{Y}
 \end{aligned} \quad (16)$$

where A_s, B_s , and C_s are constants determined by the boundary conditions, and θ_s denotes the modal (natural) frequencies, and

$$\delta_s = \frac{l_s}{L_x}, \delta_{ys} = \frac{l_s}{L_y} \quad (17)$$

For a single mass, equation (14) can be rewritten as

$$\begin{aligned} & \ddot{w}_p(t) + \theta_p^2 w_p(t) - \frac{1}{\psi^*} \sum_{a=1}^{\infty} [R_0 \eta_0 \ddot{w}_a(t) - \frac{2\sigma}{\varphi} \eta_1 w_a(t) - \frac{D_x}{\varphi} \eta_2 w_a(t) - \frac{D_y}{\varphi} \eta_3 w_a(t) \\ & + (\theta_a^2 - \frac{K_0}{\varphi}) \eta_4 w_a(t) + \frac{G_0}{\varphi} \eta_5 w_a(t) - \varsigma \varrho ((\eta_6 + \frac{1}{\pi^2} (\sum_{r=1}^{\infty} \rho_1^* \frac{\cos(2r+1)\pi vt}{2r+1} - \\ & - \sum_{r=1}^{\infty} \rho_2^* \frac{\sin(2r+1)\pi vt}{2r+1}) (\sum_{f=1}^{\infty} \rho_3^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_4^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi} (\sum_{r=1}^{\infty} \rho_5^* \frac{\cos(2r+1)\pi vt}{2r+1} - \sum_{r=1}^{\infty} \rho_6^* \frac{\sin(2r+1)\pi vt}{2r+1}) + \frac{1}{4\pi} (\sum_{f=1}^{\infty} \rho_7^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_8^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) \\ & \frac{\sin(2f+1)\pi \epsilon}{2f+1})) \ddot{w}_a(t) + 2v(\eta_7 + \frac{1}{\pi^2} (\sum_{r=1}^{\infty} \rho_9^* \frac{\cos(2r+1)\pi vt}{2r+1} - \sum_{r=1}^{\infty} \rho_{10}^* \frac{\sin(2r+1)\pi vt}{2r+1} \\ &) (\sum_{f=1}^{\infty} \rho_{11}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{12}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi} (\sum_{r=1}^{\infty} \rho_{13}^* \frac{\cos(2r+1)\pi vt}{2r+1} - \sum_{r=1}^{\infty} \rho_{14}^* \frac{\sin(2r+1)\pi vt}{2r+1}) + \frac{1}{4\pi} (\sum_{f=1}^{\infty} \rho_{15}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{16}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) \\ & \ddot{w}_a(t)) \\ & + v^2(\eta_8 + \frac{1}{\pi^2} (\sum_{r=1}^{\infty} \rho_{17}^* \frac{\cos(2r+1)\pi vt}{2r+1} - \sum_{r=1}^{\infty} \rho_{18}^* \frac{\sin(2r+1)\pi vt}{2r+1}) (\sum_{f=1}^{\infty} \rho_{19}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{20}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi} (\sum_{r=1}^{\infty} \rho_{21}^* \frac{\cos(2r+1)\pi vt}{2r+1} - \sum_{r=1}^{\infty} \rho_{22}^* \frac{\sin(2r+1)\pi vt}{2r+1}) + \frac{1}{4\pi} (\sum_{f=1}^{\infty} \rho_{23}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{24}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) \\ & \ddot{w}_a(t))] \\ & = \sum_{k=1}^{\infty} \frac{Mg}{\varphi \psi^*} \vartheta_s(vt) \vartheta_s(\epsilon) \end{aligned} \quad (18)$$

Equation (18) is the basic equation of the rectangular plate problem.

where

$$\varrho = \frac{M}{\varphi \psi^*}, \quad \varsigma = L_x L_y \quad (19)$$

$$\vartheta_s(v_j t) = \sin x_s(t) + A_s \cos x_s(t) + B_s \sinh x_s(t) + C_s \cosh x_s(t) \quad (20)$$

$$\vartheta_s(\epsilon) = \sin \omega_s + A_s \cos \omega_s + B_s \sinh \omega_s + C_s \cosh \omega_s \quad (21)$$

$$x_s = \frac{l_s v_j}{L_x}, \quad \omega_s = \frac{l_s \epsilon}{L_y} \quad (22)$$

a. Orthotropic Rectangular Plate Traversed by a Moving Force

In the moving-force approximation the inertia of the transported mass is neglected. Setting $\varrho = 0$ in (18) removes the moving-mass inertia term and (18) in a simplified form is reduced to the forced system

$$\ddot{\varpi}_p(t) + \frac{(\Gamma_p^2 - \lambda Z_6)}{[1 - \varphi \lambda R_0 \eta_0]} \varpi_p(t) + \frac{\lambda}{[1 - \varphi \lambda R_0 \eta_0]} \sum_{a=1, a \neq p}^{\infty} (\varphi R_0 \eta_0 \ddot{\varpi}_a(t) - 2\sigma \eta_1 \varpi_a(t) - D_x \eta_2 \varpi_a(t) - D_y \eta_3 \varpi_a(t) + (\varphi \theta_a^2 - K_0 \eta_4) \varpi_a(t) + G_0 \eta_5 \varpi_a(t)) = \frac{\lambda}{[1 - \varphi \lambda R_0 \eta_0]} \text{Mg} \vartheta_s(vt) \vartheta_s(\epsilon) \quad (23)$$

where

$$\Gamma_p^2 = (1 - \frac{\varphi}{\varphi^*}) \theta_p^2, \quad \lambda = \frac{1}{\varphi \eta^*}, \quad Z_6 = -2\sigma \eta_1 - D_x \eta_2 - D_y \eta_3 - K_0 \eta_4 + G_0 \eta_5 \quad (24)$$

For any arbitrary ratio λ , defined as

$$\lambda^* = \frac{\lambda}{1+\lambda}, \text{ one obtains}$$

$$\lambda = \frac{\lambda^*}{1-\lambda^*} = \lambda^* + o(\lambda^{*2}) + \dots$$

For only $o(\lambda^*)$, one obtains

$$\lambda = \lambda^*$$

Using the binomial theorem, it follows that

$$\frac{1}{1-\lambda^* \varphi R_0 \eta_0} = 1 + \lambda^* \varphi R_0 \eta_0 + o(\lambda^{*2}) + \dots \quad (25)$$

Substituting (25) into (23) and simplifying, we obtain

$$\ddot{\varpi}_p(t) + (\Gamma_p^2 - \lambda^* Z_6)(1 + \lambda^* \varphi R_0 \eta_0 + o(\lambda^{*2}) + \dots) \varpi_p(t) + \lambda^*(1 + \lambda^* \varphi R_0 \eta_0 + o(\lambda^{*2}) + \dots) \sum_{a=1, a \neq p}^{\infty} (\varphi R_0 \eta_0 \ddot{\varpi}_a(t) - 2\sigma \eta_1 \varpi_a(t) - D_x \eta_2 \varpi_a(t) - D_y \eta_3 \varpi_a(t) + (\varphi \theta_a^2 - K_0 \eta_4) \varpi_a(t) + G_0 \eta_5 \varpi_a(t)) = \lambda^* \text{Mg} (1 + \lambda^* \varphi R_0 \eta_0 + o(\lambda^{*2}) + \dots) \vartheta_s(vt) \vartheta_s(\epsilon) \quad (26)$$

Retaining only $o(\lambda^*)$ in equation (26) and simplifying further, we obtain

$$\ddot{\varpi}_p(t) + [\Gamma_p^2(1 + \lambda^* \varphi R_0 \eta_0) - \lambda^* Z_6] \varpi_p(t) + \lambda^* \sum_{a=1, a \neq p}^{\infty} [\tau R_0 \eta_0 \ddot{\varpi}_a(t) - (2\sigma \eta_1 - D_x \eta_2 - D_y \eta_3 - (\varphi \Gamma_a^2 - K_0 \eta_4) + G_0 \eta_5) \varpi_a(t)] = \lambda^* \text{Mg} \vartheta_s(vt) \vartheta_s(\epsilon) \quad (27)$$

Using the Struble technique yields

$$\Gamma_{pp} = \Gamma_p - \left[\frac{\Gamma_p^2 - Z_7}{2\Gamma_p} \right] \quad (28)$$

The equation (28) is the modified frequency for moving force problem.

where

$$Z_7 = [\Gamma_p^2(1 + \lambda^* \varphi R_0 \eta_0) - \lambda^* Z_6] \quad (29)$$

The homogeneous solution of Equation (27), using Equation (28), is obtained as

$$\ddot{\varpi}_p(t) + \Gamma_{pp}^2 \varpi_p(t) = 0 \quad (30)$$

Hence, the whole equation (27) gives

$$\ddot{\varpi}_p(t) + \Gamma_{pp}^2 \varpi_p(t) = \lambda^* Mg \vartheta_s(vt) \vartheta_s(\epsilon) \quad (31)$$

On solving equation (31) one obtains

$$\begin{aligned} \varpi_p(t) = \frac{\lambda^* Mg \vartheta_s(\epsilon)}{\Gamma_{pp}(\chi_s^4 - \Gamma_{pp}^4)} [(\chi_s^2 + \Gamma_{pp}^2)(\chi_s \sin \Gamma_{pp} t - \Gamma_{pp} \sin \chi_s t) - A_s \Gamma_{pp}(\chi_s^2 + \Gamma_{pp}^2) \\ (\cos \chi_s t - \cos \Gamma_{pp} t) - B_s(\chi_s^2 - \Gamma_{pp}^2)(\chi_s \sin \Gamma_{pp} t - \Gamma_{pp} \sinh \chi_s t) + C_s \Gamma_{pp}(\chi_s^2 \\ - \Gamma_{pp}^2)(\cosh \chi_s t - \cos \Gamma_{pp} t)] \end{aligned} \quad (32)$$

In reference to equation (5), one obtains

$$\begin{aligned} W(\mathcal{X}, \mathcal{Y}, t) = \sum_{p=1}^N \sum_{s=1}^{\infty} \frac{\lambda^* Mg \vartheta_s(\epsilon)}{\Gamma_{pp}(\chi_s^4 - \Gamma_{pp}^4)} [(\chi_s^2 + \Gamma_{pp}^2)(\chi_s \sin \Gamma_{pp} t - \Gamma_{pp} \sin \chi_s t) - A_s \Gamma_{pp} \\ (\chi_s^2 + \Gamma_{pp}^2)(\cos \chi_s t - \cos \Gamma_{pp} t) - B_s(\chi_s^2 - \Gamma_{pp}^2)(\chi_s \sin \Gamma_{pp} t - \Gamma_{pp} \sinh \chi_s t) + C_s \Gamma_{pp} \\ (\chi_s^2 - \Gamma_{pp}^2)(\cosh \chi_s t - \cos \Gamma_{pp} t)] (\sin \frac{1_s}{L\mathcal{X}} \mathcal{X} + A_s \cos \frac{1_s}{L\mathcal{X}} \mathcal{X} + B_s \sinh \frac{1_s}{L\mathcal{X}} \mathcal{X} + C_s \\ \cosh \frac{1_s}{L\mathcal{X}} \mathcal{X}) (\sin \frac{1_s}{L\mathcal{Y}} \mathcal{Y} + A_s \cos \frac{1_s}{L\mathcal{Y}} \mathcal{Y} + B_s \sinh \frac{1_s}{L\mathcal{Y}} \mathcal{Y} + C_s \cosh \frac{1_s}{L\mathcal{Y}} \mathcal{Y}) \end{aligned} \quad (33)$$

which is the transverse displacement of a moving force problem of orthotropic rectangular plate.

Orthotropic Rectangular Plate Traversed by a Moving Mass

For the moving-mass problem, interaction between the applied loads and the transported mass generates distributed inertial forces on the plate; therefore the inertial term in the governing equation must be retained (ie. $q \neq 0$). Accordingly, the full equation of motion in (18) must be retained and solved.

We replace the original operator by a modified free-system operator Γ_{pp} that encapsulates the dominant homogeneous response at a shifted frequency obtained for

moving force by Applying Struble's approximation. Consequently, equation (18) can be recast and simplified accordingly

$$\begin{aligned}
 & \ddot{w}_p(t) + 2v\varrho\Theta^*(\eta_7 + \frac{1}{\pi^2}(\sum_{r=1}^{\infty} \rho_9^* \frac{\cos(2r+1)\pi v_j t}{2r+1} - \sum_{r=1}^{\infty} \rho_{10}^* \frac{\sin(2r+1)\pi v t}{2r+1}) \\
 & (\sum_{f=1}^{\infty} \rho_{11}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{12}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi}(\sum_{r=1}^{\infty} \rho_{13}^* \frac{\cos(2r+1)\pi v t}{2r+1} - \\
 & \sum_{r=1}^{\infty} \rho_{14}^* \frac{\sin(2r+1)\pi v t}{2r+1}) + (\sum_{f=1}^{\infty} \rho_{15}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{16}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1})) \\
 & \dot{w}_p(t) + (\Gamma_{pp}^2(1 - \varrho\Theta^*(\eta_6 + \frac{1}{\pi^2}(\sum_{r=1}^{\infty} \rho_1^* \frac{\cos(2r+1)\pi v t}{2r+1} - \sum_{r=1}^{\infty} \rho_2^* \frac{\sin(2r+1)\pi v t}{2r+1}) \\
 & (\sum_{f=1}^{\infty} \rho_3^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_4^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi}(\sum_{r=1}^{\infty} \rho_5^* \frac{\cos(2r+1)\pi v t}{2r+1} - \\
 & \sum_{r=1}^{\infty} \rho_6^* \frac{\sin(2r+1)\pi v t}{2r+1}) + (\sum_{f=1}^{\infty} \rho_7^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_8^* \frac{\sin(2f+1)\pi \epsilon}{2f+1})))) + \\
 & v^2\varrho\Theta^*(\eta_8 + \frac{1}{\pi^2}(\sum_{r=1}^{\infty} \rho_{17}^* \frac{\cos(2r+1)\pi v t}{2r+1} - \sum_{r=1}^{\infty} \rho_{18}^* \frac{\sin(2r+1)\pi v t}{2r+1})(\sum_{f=1}^{\infty} \rho_{19}^* \\
 & \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{20}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi}(\sum_{r=1}^{\infty} \rho_{21}^* \frac{\cos(2r+1)\pi v t}{2r+1} - \sum_{r=1}^{\infty} \rho_{22}^* \\
 & \frac{\sin(2r+1)\pi v t}{2r+1}) + (\sum_{f=1}^{\infty} \rho_{23}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{24}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}))\ddot{w}_p(t) \\
 & + \varrho\Theta^* \sum_{a=1, a \neq p}^{\infty} [(\eta_6 + (\sum_{r=1}^{\infty} \rho_1^* \frac{\cos(2r+1)\pi v t}{2r+1} - \sum_{r=1}^{\infty} \rho_2^* \frac{\sin(2r+1)\pi v t}{2r+1})(\sum_{f=1}^{\infty} \\
 & \rho_3^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_4^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi}(\sum_{r=1}^{\infty} \rho_5^* \frac{\cos(2r+1)\pi v t}{2r+1} - \sum_{r=1}^{\infty} \rho_6^* \\
 & \frac{\sin(2r+1)\pi v t}{2r+1}) + (\sum_{f=1}^{\infty} \rho_7^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_8^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}))\ddot{w}_a(t) + \\
 & 2v(\eta_7 + \frac{1}{\pi^2}(\sum_{r=1}^{\infty} \rho_9^* \frac{\cos(2r+1)\pi v t}{2r+1} - \sum_{r=1}^{\infty} \rho_{10}^* \frac{\sin(2r+1)\pi v t}{2r+1})(\sum_{f=1}^{\infty} \rho_{11}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} \\
 & - \sum_{f=1}^{\infty} \rho_{12}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi}(\sum_{r=1}^{\infty} \rho_{13}^* \frac{\cos(2r+1)\pi v t}{2r+1} - \sum_{r=1}^{\infty} \rho_{14}^* \frac{\sin(2r+1)\pi v t}{2r+1}) \\
 & + (\sum_{f=1}^{\infty} \rho_{15}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{16}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}))\ddot{w}_a(t) + v^2(\eta_8 + \frac{1}{\pi^2}(\sum_{r=1}^{\infty} \rho_{17}^* \\
 & \frac{\cos(2r+1)\pi v t}{2r+1} - \sum_{r=1}^{\infty} \rho_{18}^* \frac{\sin(2r+1)\pi v t}{2r+1})(\sum_{f=1}^{\infty} \rho_{19}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{20}^* \\
 & \frac{\sin(2f+1)\pi \epsilon}{2f+1}) + \frac{1}{4\pi}(\sum_{r=1}^{\infty} \rho_{21}^* \frac{\cos(2r+1)\pi v t}{2r+1} - \sum_{r=1}^{\infty} \rho_{22}^* \frac{\sin(2r+1)\pi v t}{2r+1}) + \frac{1}{4\pi} \\
 & (\sum_{f=1}^{\infty} \rho_{23}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{24}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}))\ddot{w}_a(t)] = \Theta^* M g \vartheta_s(vt) \vartheta_s(\epsilon) \quad (34)
 \end{aligned}$$

Applying the modified asymptotic method of Struble as applied to moving force problem, equation (33) can be re-expressed for the homogeneous case as

$$\ddot{\varpi}_p(t) + Z_p^2 \varpi_p(t) = 0 \quad (35)$$

Hence, the whole equation becomes

$$\ddot{\varpi}_p(t) + Z_p^2 \varpi_p(t) = \Theta^* Mg \vartheta_s(vt) \vartheta_s(\epsilon) \quad (36)$$

where

$$Z_p = \Gamma_{pp} - \frac{1}{2\Gamma_{pp}} [\Gamma_{pp}^2 \Theta^* (\eta_6 + \frac{1}{\pi^2} (\sum_{r=1}^{\infty} \rho_1^* \frac{\cos(2r+1)\pi vt}{2r+1} - \sum_{r=1}^{\infty} \rho_2^* \frac{\sin(2r+1)\pi vt}{2r+1})) - v^2 \Theta^* (\eta_8 + \frac{1}{4\pi} (\sum_{f=1}^{\infty} \rho_{23}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{24}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1}))] \quad (37)$$

which gives the modified frequency representing the frequency of the free system.

The entire expression of equation (35) gives

$$\ddot{\varpi}_p(t) + Z_p^2 \varpi_p(t) = \Theta^* Mg \vartheta_s(\epsilon) [\sin \chi_s(t) + A_s \cos \chi_s t + B_s \sinh \chi_s t + C_s \cosh \chi_s t] \quad (38)$$

Adopting the procedures applied to solve equation (31) earlier, one obtains

$$\varpi_p(t) = \frac{\Theta^* Mg \vartheta_s(\epsilon)}{Z_p(\phi_s^4 - Z_p^4)} [(\phi_s^2 + Z_p^2)(\chi_s \sin Z_p t - Z_p \sin \chi_s t) - A_s Z_p(\chi_s^2 + Z_p^2)(\cos \chi_s t - \cos Z_p t) - B_s(\chi_s^2 - Z_p^2)(\chi_s \sin Z_p t - Z_p \sinh \chi_s t) + C_s Z_p(\chi_s^2 - Z_p^2)(\cosh \chi_s t - \cosh Z_p t)] \quad (39)$$

In reference to equation (4), one gets

$$W(x, y, t) = \sum_{p=1}^N \sum_{s=1}^{\infty} \frac{\Theta^* Mg \vartheta_s(\epsilon)}{Z_p(\phi_s^4 - Z_p^4)} [(\chi_s^2 + Z_p^2)(\chi_s \sin Z_p t - Z_p \sin \chi_s t) - A_s Z_p(\chi_s^2 + Z_p^2)(\cos \chi_s t - \cos Z_p t) - B_s(\chi_s^2 - Z_p^2)(\chi_s \sin Z_p t - Z_p \sinh \chi_s t) + C_s Z_p(\chi_s^2 - Z_p^2)(\cosh \chi_s t - \cosh Z_p t)] (\sin \frac{1_s}{L_x} \mathcal{X} + A_s \cos \frac{1_s}{L_x} \mathcal{X} + B_s \sinh \frac{1_s}{L_x} \mathcal{X} + C_s \cosh \frac{1_s}{L_x} \mathcal{X}) (\sin \frac{1_s}{L_y} \mathcal{Y} + A_s \cos \frac{1_s}{L_y} \mathcal{Y} + B_s \sinh \frac{1_s}{L_y} \mathcal{Y} + C_s \cosh \frac{1_s}{L_y} \mathcal{Y}) \quad (40)$$

the transverse displacement response to a moving mass of an orthotropic rectangular plate.

Orthotropic Rectangular Plate with Free Elastic Boundary Conditions

For free end at $x = 0$ and $y = 0$ and elastically supported at $x = L_x$ and $y = L_y$ are expressed as

$$W''(0, L_y, t) = 0 = W'''(0, L_y, t) \quad (41)$$

at the end $X = 0$ and

$$W''(L_x, L_y, t) - \varphi_1 W'(L_x, L_y, t) = 0 = W'''(L_x, L_y, t) + \varphi_2 W(L_x, L_y, t) \quad (42)$$

at the end $x = L_x$

In the same vein, we have

$$W''(L_x, 0, t) = 0 = W'''(L_x, 0, t) \quad (43)$$

at the end $y = 0$ and

$$W''(L_x, L_y, t) - \varphi_1 W'(L_x, L_y, t) = 0 = W'''(L_x, L_y, t) + \varphi_2 W(L_x, L_y, t) \quad (44)$$

at the end $y = L_y$

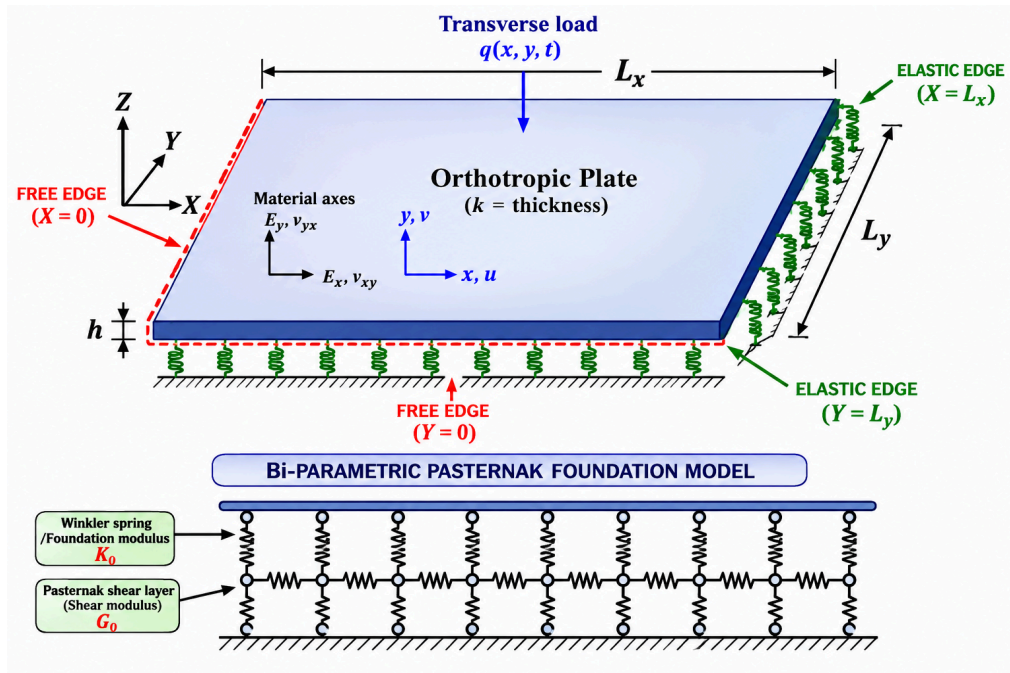


Figure 3: Representation of Free Edge Orthotropic Rectangular Plate

Thus, for normal modes, we have

$$\vartheta_s''(0) - \varphi_1 \vartheta_s'(0) = 0 = \vartheta_s'''(0) + \varphi_2 \vartheta_s(0) \quad (45)$$

at the end $x = 0$ and $y = 0$ and

$$\begin{aligned} \vartheta_s''(L_x) - \varphi_1 \vartheta_s'(L_x) &= 0 = \vartheta_s'''(L_x) + \varphi_2 \vartheta_s(L_x) \\ \vartheta_s''(L_y) - \varphi_1 \vartheta_s'(L_y) &= 0 = \vartheta_s'''(L_y) + \varphi_2 \vartheta_s(L_y) \end{aligned} \quad (46)$$

Using equations (45) and (46), it can be shown that

$$A_s = \frac{\frac{\iota_s}{L_x} [\sinh \sinh \iota_s - \sinh \sin \iota_s] + \varphi_1 [\cosh \cosh \iota_s + \csc \cos \iota_s]}{\frac{\chi_s}{L_x} [\cosh \cosh \chi_s - \csc \cos \chi_s] - \varphi_1 [\sinh \sinh \chi_s - \sinh \sin \chi_s]} = \frac{\frac{\iota_s^3}{L_x^3} [\csc \cos \iota_s - \cosh \cosh \iota_s] - \varphi_2 [\sinh \sinh \iota_s + \sinh \sin \iota_s]}{\frac{\iota_s^3}{L_x^3} [\sinh \sinh \iota_s + \sinh \sin \iota_s] + \varphi_2 [\csc \cos \iota_s + \cosh \cosh \iota_s]} = C_s \quad (47)$$

and

$$B_s = 1 \quad (48)$$

By solving equation (46), we obtain the characteristic (frequency) equation governing the dynamics of the system

$$\tan \iota_s = \tanh \iota_s \quad (49)$$

Hence, we have

$$\iota_1 = 3.927, \iota_2 = 7.069, \iota_3 = 10.210, \dots \quad (50)$$

Substituting Eqs. (46), (47) and (48) into Eqs. (32) and (39) produces the closed-form modal displacement expressions that describe the plate's dynamic response to both a moving force and a moving mass on a Pasternak elastic foundation.

II. RESULTS AND DISCUSSION

A. Comments on the Closed-Form Solution

To study the resonance phenomenon in this conservative system, equation (32) that shows the orthotropic rectangular plate with elastically restrained ends subjected to a distributed load moving at constant speed attains resonance whenever the forcing frequency associated with the moving load coincides with one of the system's modal (natural) frequencies

$$\chi_s = \Gamma_{pp} \quad (51)$$

while equation (39) that shows the elastically-restrained orthotropic plate on a Pasternak foundation subjected to a distributed mass moving at constant speed reaches resonance whenever the forcing frequency associated with the moving mass equals the inertia-shifted modal frequency of the system.

$$\chi_s = Z_p \quad (52)$$

where

$$Z_p = \Gamma_{pp} - \frac{1}{2\Gamma_{pp}} [\Gamma_{pp}^2 \mathbf{Q} \Theta^*(\eta_6 + \frac{1}{\pi^2} (\sum_{r=1}^{\infty} \rho_1^* \frac{\cos(2r+1)\pi vt}{2r+1} - \sum_{r=1}^{\infty} \rho_2^* \frac{\sin(2r+1)\pi vt}{2r+1})) - v^2 \mathbf{Q} \Theta^*(\eta_8 + \frac{1}{4\pi} (\sum_{f=1}^{\infty} \rho_{23}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{e=1}^{\infty} \rho_{24}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1})))] \quad (53)$$

Comparing equations (51) and (52), we obtain

$$Z_p = \Gamma_{pp} [1 - \frac{1}{2\Gamma_{pp}^2} (\Gamma_{pp}^2 \mathbf{Q} \Theta^*(\eta_6 + \frac{1}{4\pi} (\sum_{f=1}^{\infty} \rho_7^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_8^* \frac{\sin(2f+1)\pi \epsilon}{2f+1})) - v^2 \mathbf{Q} \Theta^*(\eta_8 + \frac{1}{4\pi} (\sum_{f=1}^{\infty} \rho_{23}^* \frac{\cos(2f+1)\pi \epsilon}{2f+1} - \sum_{f=1}^{\infty} \rho_{24}^* \frac{\sin(2f+1)\pi \epsilon}{2f+1})))] = \Gamma_{pp} \quad (54)$$

B. Discussion of Simulated Figures

In order to discuss the analysis presented, we consider an orthotropic rectangular plate of length $L_y = 0.923m$ and breadth $L_x = 0.432m$, load velocity $c = 0.8123m/s$ and $\epsilon = 0.4m$ with elastically-restrained free edges. Results are presented as plotted response curves for the elastically-restrained free edges: time-domain deflections, modal contributions, and parametric variation studies (rotatory-inertia correction factor, Winkler modulus, and Pasternak shear modulus). All dimensional quantities are stated in SI units; where useful, nondimensional forms of the responses are also reported to aid comparison across geometries and parameter sets.

a. Effect of Rotatory Inertia Correction Factor

Rotary inertia adds extra resistance to angular acceleration. Increasing it, leads to stiffening behavior. Figures 3(a) and 3(b) depict the response amplitudes decrease as the value of rotatory inertia correction factor R_o increases for both cases of moving distributed forces and moving distributed masses respectively.

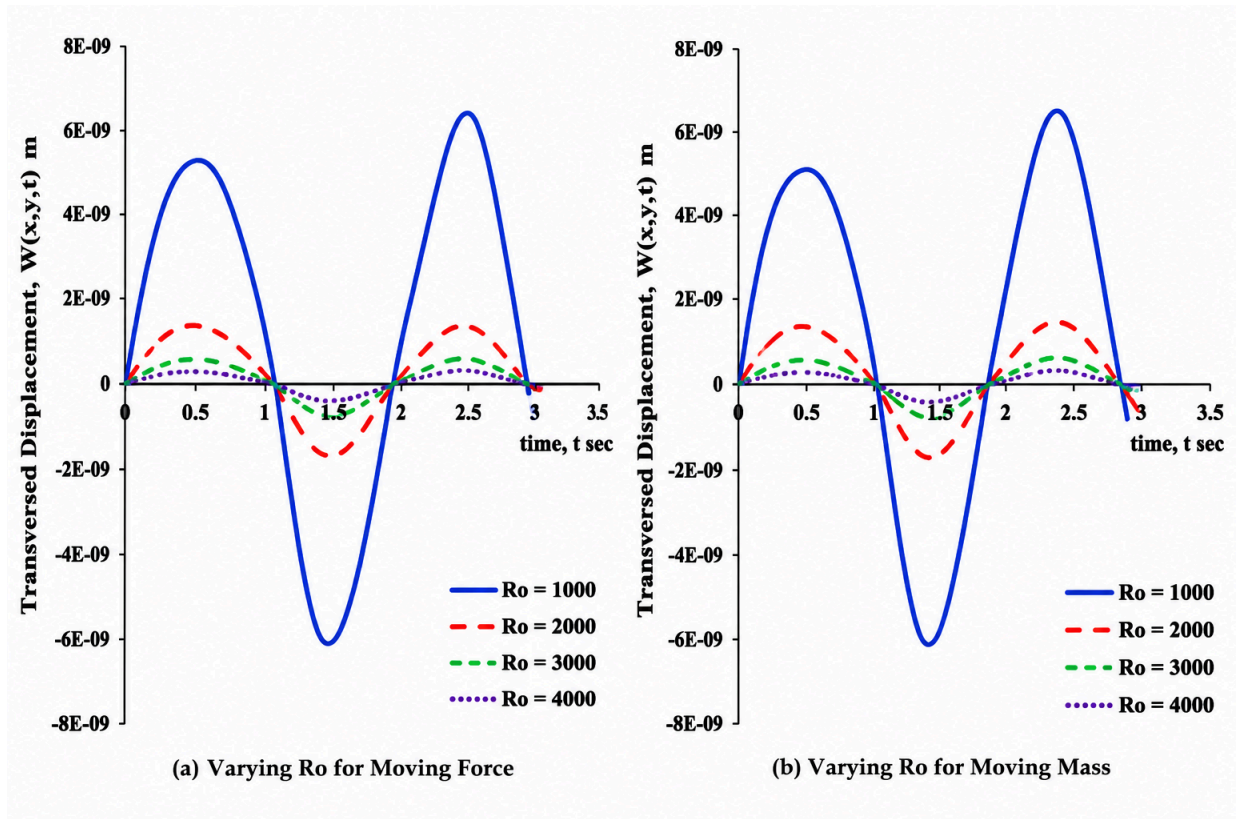


Figure 3: Effect of R_o on Displacement Profile of the Plate

b. Effect of Foundation Modulus

A low K_o offers little vertical support, so the plate deflects significantly under the moving mass. As K_o increases, the springs resist vertical displacement more strongly, effectively stiffening the support. This reduces the plate's mid-span deflection for both cases of moving distributed forces and moving distributed masses as shown in figures 4(a) and 4(b) respectively.

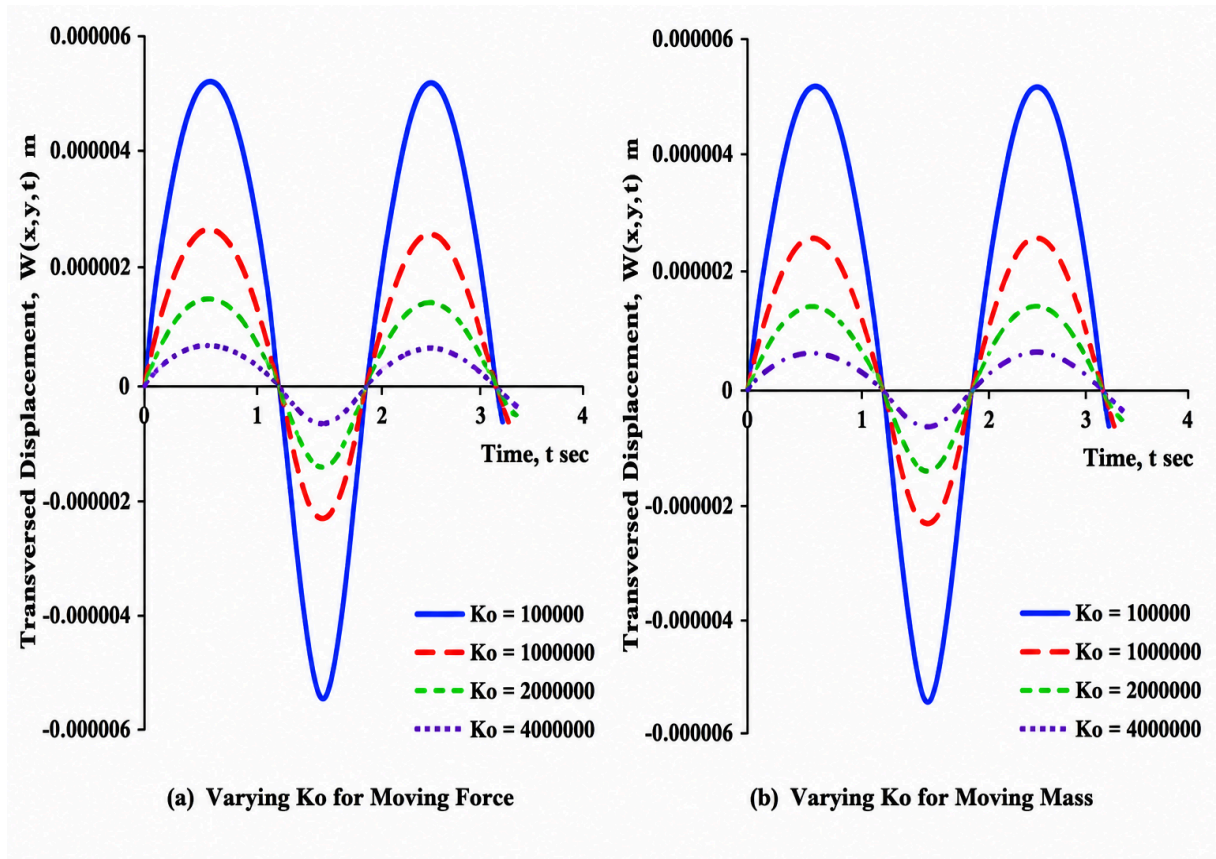


Figure 4: Effect of K_o on Displacement Profile of the Plate

c. Effect of Shear Modulus

This measures the plate material's resistance to shear deformation. With no shear modulus (G_o), mid-span deflection is highest. As G_o grows, each successive curve shows a reduced peak for both cases of moving distributed forces and moving distributed masses as depicted in figures 5(a) and 5(b), respectively.

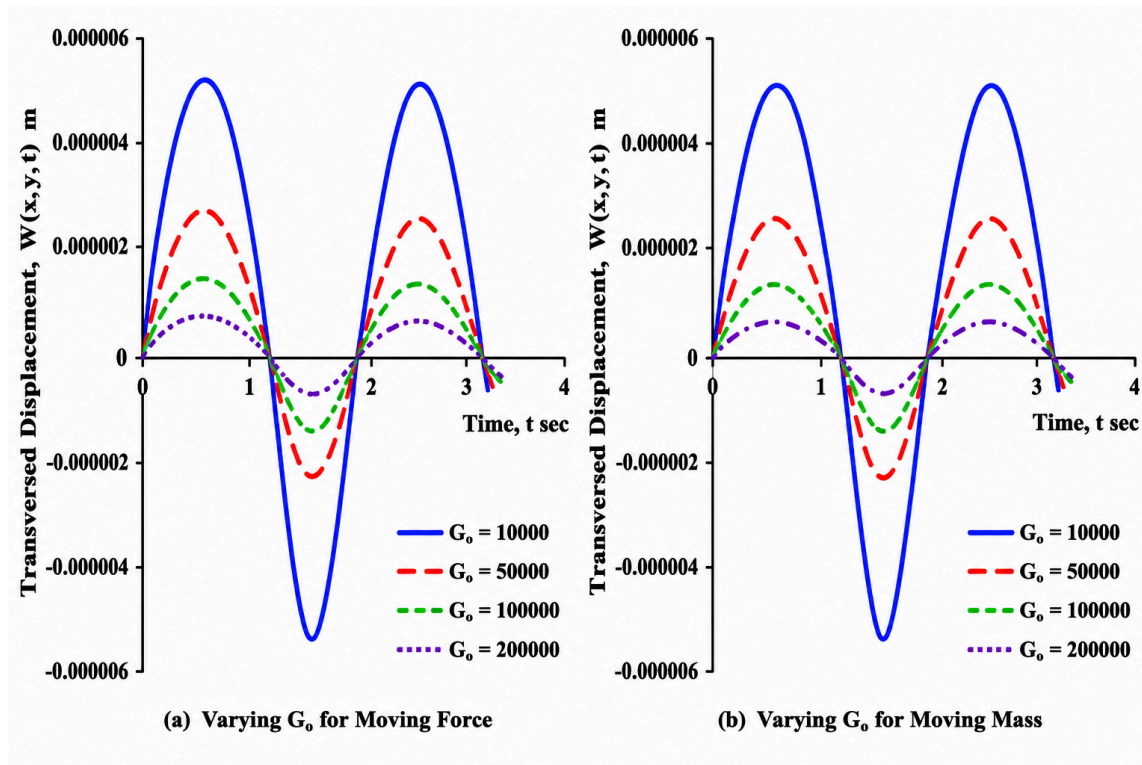


Figure 5: Effect of G_o on Displacement Profile of the Plate

IV. ENGINEERING APPLICATIONS

The mathematical rigor of the proposed framework extends beyond formulation to provide a predictive and design-oriented basis for vibration analysis and control of orthotropic plate foundation systems. By explicitly accounting for foundation stiffness, boundary restraint, and plate foundation interaction, the model enables a systematic assessment of their coupled influence on structural dynamic performance. Its principal engineering value lies in its capability to resolve accurate transient deflection histories and maintain phase fidelity under sustained oscillatory excitation. Consequently, the framework provides a robust approximate analytical model for a plate on an elastic foundation under moving distributed masses. The framework also provides vibration control-oriented design of plate structures interacting with elastic foundations. This model has meaningful applications in engineering in the areas of composite floor systems, Aircraft panels, Foundation-supported slabs, Smart floor vibration control, etc.

V. CONCLUSION

This study has developed closed-form analytical solutions for the dynamic response of elastically restrained orthotropic rectangular plates resting on a Pasternak two-parameter foundation and subjected to moving distributed forces and moving distributed masses. The singular fourth-order governing partial differential equation was regularized following the approach of Shadnam et al. (2001) and systematically reduced to a coupled system of second-order ordinary differential equations through a generalized Galerkin formulation. A modified Struble asymptotic procedure, combined with Laplace-transform analysis,

subsequently yielded explicit modal expressions for the transient plate response and enabled direct identification of the associated resonance conditions.

The analytical formulation provides a unified framework for quantifying the coupled effects of orthotropy, asymmetric elastic restraint, foundation interaction, moving-load velocity, and transported-mass inertia. In particular, the results demonstrate that the Winkler stiffness and Pasternak shear parameter provide distinct mechanisms for modifying the dynamic characteristics, while rotatory inertia can become important in accurately describing the response, particularly when higher-frequency modal effects are relevant. The study further establishes that the moving-mass and moving-force idealizations are not dynamically interchangeable: inclusion of the inertia and inertial coupling of the transported mass can lead to appreciably different resonance characteristics, modal amplitudes, and transient responses. Consequently, the moving-force approximation should be applied with caution when the mass of the moving system is non-negligible relative to the supporting structure

Future work should extend the present analytical framework to include material and geometric nonlinearity, viscous or frequency-dependent damping, variable (non-constant) foundation parameters, and validation against high-fidelity finite-element simulations and experimental measurements. Such extensions will further quantify the practical limits of the moving-force idealization and support robust vibration-mitigation strategies for real structures.

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CONFLICT OF INTEREST

The authors declared that there is no conflict of interest.

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